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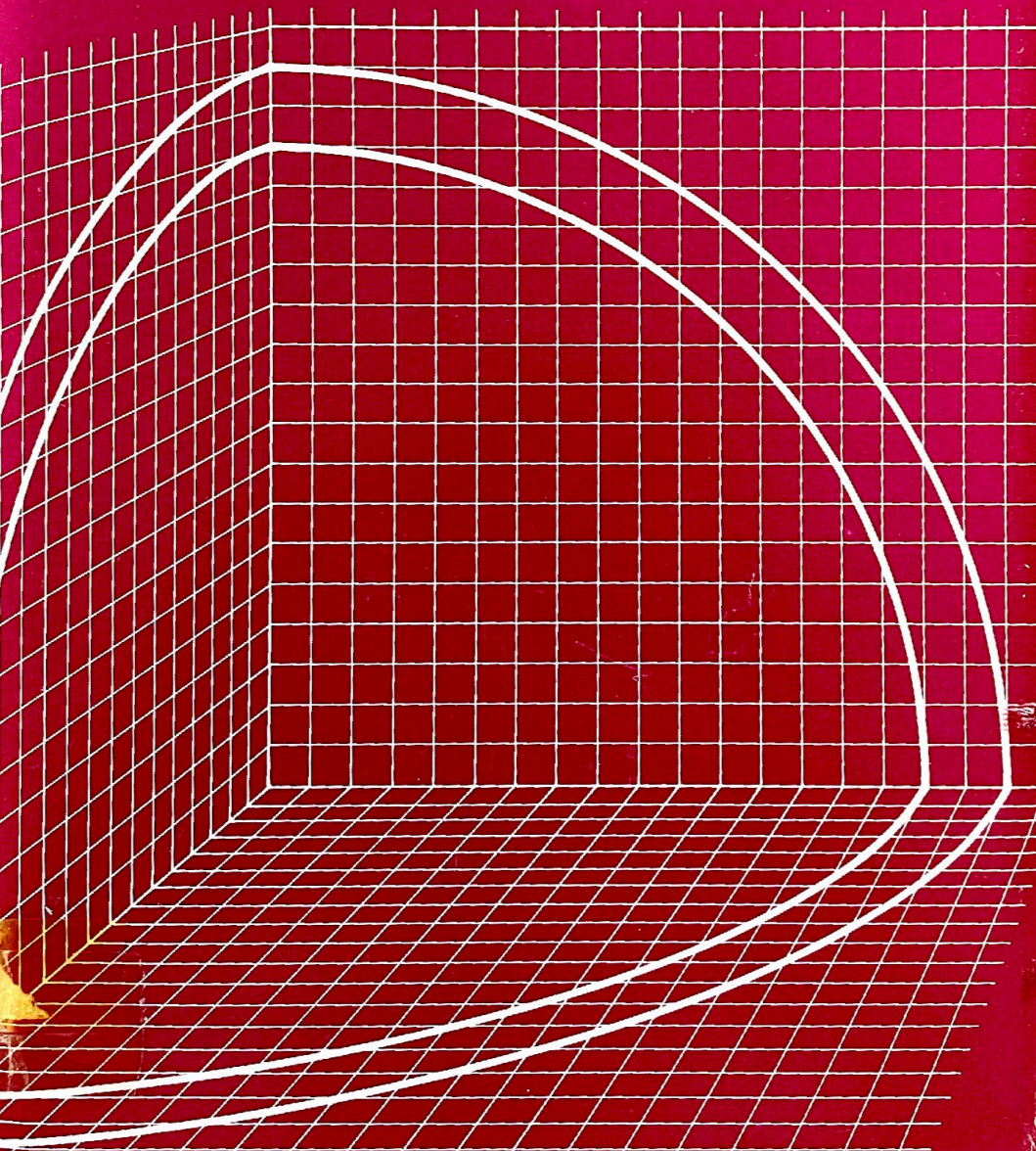
THE PROPERTIES OF NUCLEI

G.A. Jones



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Editors' foreword

This book fits closely into the network of topics of the Oxford Physics Series and at the same time preserves an integrity of its own. The earlier complementary core texts include *Radiation and quantum physics*, and *Atomic nuclei and their particles* with supporting texts on *Electromagnetism* and *Special relativity*, and *Atoms and their structure*.

The volume *Basic quantum mechanics* will provide the necessary introduction to many sections of this new text on *The properties of nuclei*, the level of which is pitched to take over where the corresponding sections of *Atomic nuclei and their particles* leaves off.

Dr. Jones concentrates his attention on the available data and their interpretation, developing a natural sequence from the properties of nuclear forces, through nuclear models to nuclear decay and reactions including fission and fusion. The study of these topics is undertaken in second- or third-year courses in physics, applied physics, and nuclear engineering and challenges the student to appreciate both the power of the available experimental and theoretical methods and also the wide range of opportunities for research in the nuclear many-body problem. The sureness of approach found in this treatment illuminates the scene in a rewarding fashion which reflects the author's experience of teaching and developing the subject over a number of years.

March 1977. E.J.B.

Preface

The level I have aimed at in this book is that of the nuclear-structure course given to Oxford students in the final term of their second year. In their third year some, but not all, pursue a more advanced course, according to their choice of options. The book therefore reflects the requirements of 'mainstream' undergraduate physics courses. It presupposes a certain familiarity with particle accelerators and radiation detection.

The content is chiefly devoted to the static and quasi-static properties of nuclei and nuclear excited states. But I firmly believe that all physics students should have some notion of how a nuclear reactor works, and so a chapter on nuclear reactions has been included in addition to a brief account of fission (and fusion).

In a book of this length much has to be taken on trust; quantum-mechanical formulae have been used freely with no more than a passing reference (the particular reference is to a text which I found useful as a student and still do as a teacher, but a more recent publication may appeal to the reader). Whatever the choice, I hope my book will encourage the student to acquire a working knowledge of basic quantum mechanics, which forms the framework of his (her) studies.

I am indebted to members of the Department of Nuclear Physics, Oxford, who read and commented on the script, and especially to Dr. M.G. Bowler with whom I had many valuable discussions. I should also like to thank one of the editors of this series (E.J.B.), who,

I feel, put more time than his editorial duties demanded into a critical appraisal which I much appreciated.

Finally, a number of the figures were taken from, or based on, published work. The figure captions indicate, where relevant, the author and the work. My thanks are also due to the publishers for permission to reproduce published work: They are: Fig. 1.2, Springer Verlag; Figs. 2.1, 3.8, 3.9, 4.3(a),(b), North Holland; Figs. 2.4, 6.3, American Institute of Physics; Fig. 3.2, Pergamon Press; Fig. 3.3, Annual Reviews; Fig. 3.4, Academy Munksgaard; Fig. 4.2, McGraw Hill; Fig. 5.2, John Wiley and Sons; Fig. 6.4, Brookhaven National Laboratory; Fig. 7.3, W.A. Benjamin.

G.A.J.

Contents

1. General properties of nuclei

INTRODUCTION

From early measurements on Coulomb scattering it is known that the atomic nucleus is a small central particle (radius $\ll 10^{-11}$ cm) which possesses virtually the whole mass of the atom and which carries a positive charge equal to the atomic number of the atom. From later work it is known that the nucleus is built up of protons and neutrons; the former supply the positive charge, whilst both contribute to the mass. From general quantum-mechanical considerations such a confined system will possess energy levels characterized by quantum numbers. Under normal circumstances, the nucleus will be in its ground state, and this chapter is concerned with nuclear properties in this state. To determine the properties of the nucleus in other (excited) states it is necessary to set up such states by means of nuclear reactions.

Thus a particular nucleus (in its ground state) must be described in terms of the quantum numbers 'angular momentum' (also loosely called spin) and 'parity' (to be explained later in the chapter). This is the minimum number of quantum numbers required by the symmetry properties of the system. There are others, but they are approximate or model-dependent, i.e. they depend on a simplified picture of the nucleus, and, unlike the corresponding atomic system, no one model stands out sufficiently to attach extra significance to its quantum numbers.

In addition, a system of finite extent can be des-

cribed in terms of moments of the matter distribution. In the case of the nucleus these moments are usually determined from its interaction with the atomic electrons which is electromagnetic in origin, so the moments measured are electric and magnetic multipole moments. A system can have an electric dipole moment only if it has an up-down directional asymmetry in space, e.g. the molecule HCl, but not Cl_2, can have a permanent electric dipole moment. For nuclei no such asymmetry is believed possible, so nuclei do not possess such a moment. Since the components of the nucleus, protons and neutrons, have intrinsic spins and magnetic dipole moments, the nucleus itself can also have a magnetic moment. The two important moments of the nucleus, apart from the electric monopole, are therefore its magnetic dipole and electric quadrupole moments.

Whilst nuclear physics deals with all nuclei, of especial interest are the so-called 'stable nuclei' These are the nuclei which exist in the world around us, and this is quite specific, though 'stable' itself is much less definite. A nucleus may be unstable against emission of a neutron or a proton, or a composite particle such as an α-particle, or against β-decay. Of these neutron emission is very fast ($\tau \sim 10^{-16}$ or less), proton emission generally less so but still very fast, whilst the latter two can have lifetimes varying over a vast range depending on the energy available ($\tau \sim \mu s$ to $\sim 10^{20}$ years). By 'stable' a lifetime of the order of the age of the universe ($\sim 10^{10}$ years) is implied; sometimes it has the restricted meaning 'stable against the β-decay process'.

THE SIZE

The original measurements made by Rutherford and

his co-workers have now been augmented by more accurate work, which has not only measured an effective radius for the nucleus but has also obtained a measure of the rate of fall-off of density, or charge, at the periphery. The measurements fall into two classes: (1) those that determine the charge distribution and (2) those that determine the matter distribution.

The charge distribution

These measurements are made with charged leptons as probes. Leptons form a class of fundamental particles which interact only weakly with protons and neutrons, except for the long-range Coulomb term where relevant.† Thus the interaction between an electron or a muon and nuclear matter is completely dominated by the Coulomb interaction. The greatest source of information is the elastic scattering of electrons. Accurate measurements of optical and X-ray spectra and muonic X-ray spectra have also given information on the nuclear size.

Electron elastic scattering

Outside the nucleus of charge Ze the electron experiences a force $\propto Z/r^2$, but inside the nucleus (assumed spherical) the force is $\propto Z'(r)/r^2$, where $Z'(r)$ is the charge contained in the sphere of radius r. Thus small-angle scattering, which corresponds to

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†Interactions between fundamental particles can be put into the categories strong, electromagnetic, weak and gravitational, which are roughly in strength as $10 : 1/137 : 10^{-7} : 10^{-39}$ (see e.g. Perkins 1972).

a classical trajectory rather distant from the nucleus, will obey the Rutherford formula and therefore give no information about the distribution of charge, whilst large-angle scattering will show a significant deviation from the formula. The ratio to Rutherford scattering as a function of angle is determined by the form of $Z'(r)$ and therefore by the charge distribution. From the uncertainty principle, if we wish to attach significance to changes of density over a distance Δr, then the incident particle must be capable of transferring momentum $\hbar q$ to the target nucleus such that $q\Delta r \sim 1$; obviously the wavenumber of the incident beam must exceed q. For $\Delta r \sim 10^{-13}$cm, electrons of kinetic energy > 200 MeV are required. Some twenty years ago, measurements were made at 50 MeV and were capable of defining a suitably averaged charge radius only; more recently, the energy range has been extended to 600 - 1000 MeV and the radial dependence of the nuclear charge distribution has been revealed in fine detail.

X-ray and optical spectra

Since the nucleus has finite extent its field of force on the atomic electrons will not be $\propto 1/r^2$ when the electron is within the nucleus. If therefore the atomic energy levels can be calculated accurately for a point nucleus and compared with their measured values, the nuclear size can be deduced. But the calculation is an unsolved many-body problem. For X-ray transitions the screening effect of the outer electrons is not large and can be approximated to; but for the tightly-bound inner electrons there are uncertainties even in the solution of the two-body problem, whilst for the valence electrons, although these latter un-

certainties do not arise, the screening effects cannot be calculated to sufficient accuracy. In certain cases useful information can be derived. It is reasonable to assume that, for a comparison between spectral lines from isotopes of the same element, the differences will arise from the differences in nuclear mass and size only. For light elements the mass difference produces effects large enough to measure, but for heavier elements the measured effects must be ascribed to differences in nuclear charge distribution. Thus the method of isotope shift gives information on the way the charge distribution changes on adding neutrons to a nucleus.

Muonic X-rays

Negative muons μ^-, like electrons, can be captured into atoms and interact electromagnetically with nuclei. Since they have a mass $\sim 200\ m_e$ (where m_e is the mass of the electron), their Bohr orbits will have 1/200th of the radius of the equivalent electron orbits. For a heavy element such as lead, the lowest Bohr orbit (since no orbits are occupied by other muons, the exclusion principle does not operate here) is such that the muon spends most of its time within the nucleus. The size effect is therefore large, reducing the lowest $p \rightarrow s$ transition from ~ 16 MeV for a point nucleus to ~ 6 MeV as measured. The muon is, of course, unstable and decays spontaneously to an electron and two neutrinos (ν) with a mean life of $\sim 2\ \mu s$. In the lowest Bohr orbit in lead it decays in about half this time because of the competing mode $\mu^- + p \rightarrow n + \nu_\mu$, but during its lifetime it traverses several metres of solid nuclear matter (density $\sim 3 \times 10^{17}$ kg m^{-3}), thus

emphasizing the weakness of the weak interaction.

From a theoretical aspect, the computational difficulties are the same as those encountered in interpreting electronic X-rays, but the uncertainties are reduced. Obviously the screening effects of the atomic electrons will only be of the same order as for the K-electrons, whereas the nuclear interaction energy is increased by the factor 200. It also turns out that the uncertainties in the two-body problem are reduced by the same factor. Since the bound muons have effectively a long wavelength ($\sim$ size of the Bohr orbit) the measurements lead to an average nuclear radius and give no information on the charge distribution. But the average they give is very precise, and nowadays it is used as a datum in the analysis of electron scattering, acting as a constraint when looking for the finer details of the charge distribution.

The matter distribution

If the nuclear probe interacts strongly with protons and neutrons then the size determined from the interaction will be that of the nuclear matter as a whole. We have also the added complication that the strong interaction has a range which, though short, is not short compared to the nuclear size, and due allowance must be made for this interaction when deducing the mass distribution.

The historical example of this kind of measurement is the elastic scattering of α-particles, which led to the first determinations of the nuclear size. Other charged particles have since been used, e.g.

protons, carbon and oxygen nuclei (in the form of partially stripped ions), and other heavy ions. From a theoretical point of view, the simplest process is the scattering of neutrons, but experimentally this is rather difficult, since it involves three nuclear interactions - one to produce the neutron, another to scatter it, and a third to detect it. Other methods involving the electrostatic term are of historical importance: they are the lifetime for α-radioactivity and the determination of the electrostatic energy of the nucleus from the energy difference between mirror pairs. Recently measurements of K-mesic X-rays have proved to be a promising method of probing the nuclear surface.†

Scattering of charged particles

Experimentally, the scattering cross-section (see Appendix A) is measured as a function of angle for a fixed beam energy or as a function of energy at a fixed angle. In either case the ratio of measured cross-section to Rutherford cross-section is plotted as a function of the variable. The ratio will be unity over a range of angles from near zero up to a certain angle, beyond which it drops off, indicating absorption of the incident particle. Alternatively, for constant angle the ratio will be unity up to a certain energy, beyond which it drops off. However, for protons or even α-particles on light nuclei it is possible for the cross-section to rise above the Rutherford value,

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†K-mesons, which can be positively or negatively charged or neutral, have a mass $\sim 1000\ m_e$ and interact strongly with nucleons.

because such particles stand a good chance of being emitted after absorption. For heavier ions, or protons and α-particles on heavy nuclei, emergence of the absorbed particle is improbable.

The region of fall-off is correlated with the classical trajectory passing through the periphery of the nucleus, immediately giving an idea of the size of the nucleus. How to obtain a more precise determination of the size was uncertain before the development of the optical model (see Chapter 3). We should notice that, in the interpretation of the results, in addition to making allowance for the range of nuclear forces it is also necessary to take into account the finite size of the incident particle.

Neutron scattering

When fairly energetic neutrons ($\sim$ 10 - 30 MeV) impinge on nuclei they are strongly absorbed from the initial beam, though they may subsequently be emitted from the resulting excited nuclei. The degree to which they are emitted depends upon the number of competing modes of decay (see Chapter 6). If there are many such modes, and there are for $E \sim 10$ - 30 MeV, elastic scattering will be reduced to a minimum. Elastic scattering cannot be zero since an extension of Babinet's theorem in optics to this nuclear wave problem indicates that absorption without scattering is impossible. This scattering is predominantly through small angles (up to $1/kR$ as in optical diffraction). The absorption cross-section can be conveniently measured by a beam-attenuation technique. A neutron detector placed in the path of a collimated neutron beam determines the beam flux before and after interposition of

the absorbing target. If the angle subtended by the detector at the target is greater than the angle of diffraction, then diffracted neutrons are detected, and the beam attenuation is a true measure of nuclear absorption. A simple theory gives $\sigma_{abs} = \pi(R + \lambda\!\!\!^{-})^2$, where the additional term $\lambda\!\!\!^{-}$, the wavelength of the projectile, expresses the uncertainty with which we can locate a neutron. A more sophisticated analysis can be carried out in terms of the optical model.

α-Radioactivity

From consideration of the mass equation (see pp. 12-20), all nuclei above $A \sim 150$ are unstable, in principle, and can break up into two smaller units. Because of its high intrinsic binding, the α-particle is usually one of these units. The process involved will be discussed in more detail in Chapter 4, but for the present it can be summarized by stating that the probability of α-particle emission depends on two factors: the intrinsic nuclear factor and the barrier penetrability. The latter is a rapidly varying function of the nuclear radius, which can therefore be determined if the former is known - the measured lifetime is known with great precision. Unfortunately, the intrinsic factor is a notional one and cannot be measured; theoretical estimates are uncertain, with the result that the nuclear radius deduced is qualitative only. It is probably more meaningful to use the data, and a nuclear radius from other data, to deduce the intrinsic nuclear factor.

Electrostatic energy differences of mirror nuclei

Mirror nuclei are nuclei in which the proton num-

ber and neutron number are interchanged, e.g. ^{13}C and ^{13}N having $Z = 6$, $N = 7$ and $Z = 7$, $N = 6$. Most of the measurements refer to mirror pairs having $|N - Z| = 1$.

It is believed (Chapter 2, p.42) that the specifically nuclear interactions will be the same for such pairs, so their mass difference will arise from the electrostatic interaction and the n-p mass difference. The former term is discussed in greater detail in the next section, for the moment a uniform distribution of charge in the nucleus is assumed to give for the mass difference

$$\Delta M = M(A,Z+1) - M(A,Z) = m_p + m_e - m_n + \frac{3}{5}\frac{1}{4\pi\epsilon_0}\frac{(2Z+1)e^2}{R},$$

where the M and m are defined in the next section.

The mass difference can be measured accurately from the β^+-decay spectrum of $(A, Z+1)$, since

$$\Delta M = m_e + E_{kin}$$

where E_{kin} is the maximum kinetic energy available to the β^+; or from the threshold energy for neutron production by protons on (A,Z),

$$M(A,Z) + m_p + m_e = M(A,Z+1) + m_n + Q$$

$$\Delta M = m_p + m_e - m_n - Q$$

Thus the Q-value of the (p,n) process is the negative of the electrostatic energy difference.

From the data, $R = R_0 A^{1/3}$ is found to hold good, but the value deduced for R_0, namely 1·46 fm is rather large compared with deductions from other processes

(see the end of this section). This reflects the dubious supposition that the nuclear charge is uniformly distributed within the nucleus. Since the data on mirror nuclei refer only to light nuclei ($A < 40$) surface effects will be expected to be large.

K-mesic X-rays

From the discussion of muonic X-rays, it follows that the K^--mesic Bohr orbits will have an even smaller radius. In addition, the K^--meson will interact strongly with the nucleus, so nuclear absorption will compete with X-ray emission. Absorption will occur from highly excited Bohr orbits when the K-meson overlaps the tail of the matter distribution. These measurements therefore focus attention on the details of the matter distribution at the surface of the nucleus.

Conclusions on size

Analyses of the earlier data, which were capable only of defining an effective nuclear radius, gave the result that the radius is proportional to the cube root of the mass number A, i.e. that the nuclear density is a constant independent of the nuclear size. Later measurements indicate that, although this is true for the central region, near the periphery the density falls off smoothly and quickly to zero. The precise nature of the fall-off is not determined from the data, but it has proved convenient to represent it by the (Fermi) function

$$(\rho)r \propto [1 + \exp \{\frac{(r-R)}{a}\}]^{-1},$$

where R, the radius at half-density, is referred to as

the nuclear radius and a is a surface thickness parameter which defines the region over which the density is falling off. This latter parameter appears to be independent of nuclear size, whilst $R \propto A^{1/3}$.

The measurements referring to the charge distribution are of greater precision than those referring to the matter distribution, but both agree on the values of R and a; there appears to be no significant difference between the distributions of neutrons and protons within the nucleus. The data may be summed up by giving the charge radius as $R = R_0 A^{1/3}$ where $R_0 = 1{\cdot}1$ fm and $a \sim 0{\cdot}5$ fm. The latter figure is not known accurately. The matter distribution will have the same parameters, but both are less accurate.

An objection may be raised, by the more discerning student after reading on, that nuclei have been treated as spherical whereas some of the more massive nuclei have quite large quadrupole moments. The reason is that the nuclei have not been aligned in their targets during the experiments described, and so the results give a time- and space-averaged picture of the nucleus which will have spherical symmetry even when it is intrinsically non-spherical. The averaging process may well be expected to produce a wider region of fall-off i.e. a greater value for a. There appears to be some indication of this.

THE MASS AND STABILITY

Historically, the masses of neutral atoms were determined relative to (atomic) $^{16}O \equiv 16$ atomic mass units and later relative to (atomic) $^{12}C \equiv 12$ U (unified atomic mass unit), by chemists using chemical reactions. The measurements were later improved upon

using physical techniques involving the trajectories of ionized atoms through electric and magnetic fields. With this information available the nuclear physicist found it expedient to use atomic masses in his equations for nuclear reactions, since conservation of charge ensures that the same number of electrons will appear on each side of an equation (though not for β^+-decay, see p. 21). Balancing the electron number, however, does not allow completely for the difference between atomic and nuclear masses, since no account has been taken of the binding energy of the atomic electrons - which, though small, can add up to $\sim$ 0·5 MeV for uranium. If this appeared directly as an error in the determination of Q-values in nuclear reactions then it would be serious. But most reactions are concerned only with small changes of A and Z and therefore only with small changes in this term, which, if neglected, could introduce errors of order 20 keV for α-decay in the region of uranium when using nuclear masses - though the correction is well known and accurate to a fraction of 1 keV. On the other hand, use of atomic masses largely eliminates such effects since the mass measurements are done on ions which are ionized in the valency shell and thus already include all the large contributions to electronic binding. For this reason the atomic scale is to be preferred. However, the student may come across the usage of nuclear mass (Oxford University Examination Papers are a case in point), and so, although the atomic scale will be employed here, comparisons will be made between the two scales. The nomenclature employed will be that M will refer to atomic masses and m will refer to nuclear masses. Since the masses quoted for the pro-

ton and neutron are usually m_p and m_n, rather than M_H ($M_n = m_n$ anyway), the mass equations will contain $Z(m_p + m_e)$ rather than ZM_H.

From the equivalence of mass and energy, the mass of a nucleus consisting of Z protons and N neutrons ($Z + N = A$) will be less than the sum of the masses of the individual nucleons by the energy of binding (expressed in u ($1{\cdot}66 \times 10^{-27}$ kg) - it is in fact more convenient to re-express the masses in MeV and since, in u, the mass values are close to the integers A, it is usual to give the mass excess in MeV, i.e. $\{M(A,Z) - A\}$ u converted into MeV). Thus

$$M(A,Z) = Z(m_p + m_e) + (A-Z)m_n - B.$$

It is necessary therefore to consider all contributions to the binding energy. This is done in an empirical way, but guided by physical facts and theories, in the mass equation of von Weizsäcker.

From the fact that nuclear density (neglecting surface effects) is a constant a nucleus can be compared with a liquid drop, for which it is known that the binding energy (again neglecting surface effects which are usually small for liquid drops) is proportional to its volume, since all molecules deep in the liquid are in the same environment. Hence, by analogy, the leading term in the mass equation representing the binding energy will be $-\alpha A$, where α is some undefined constant. All constants introduced into the mass equation will be defined to be positive.

A second term in the binding energy also arises from the analogy with a liquid drop. The forces which keep the liquid drop together (van der Waals forces)

are short range, so that the volume binding energy term is dominated by the interactions between near neighbours. At the surface a molecule will have fewer near neighbours than a molecule in the interior and so will be less tightly bound, giving a surface-energy term and a surface tension, the former being proportional to the surface area. It results in a lower total binding of the drop and therefore appears as a positive term in the mass of the drop. Since the surface area of a sphere $\propto r^2$, and, reverting to the nucleus, $r \propto A^{1/3}$ the surface-energy term will appear as $+\beta A^{2/3}$ in the nuclear mass equation. As a large fraction ($\sim \frac{1}{2}$) of nucleons are near the surface this will be expected to be a considerable fraction of the volume term.

The analogy with a liquid drop neglects the charge on the nucleus. On attempting to remedy this by charging the drop, the analogy breaks down, since the charge migrates to the surface of the drop whereas, from the previous section, it is known that the charge density of a nucleus is proportional to the matter density. This illustrates the difficulties in making classical models of quantum-mechanical systems; obviously the uncertainty principle operates to reduce the energy when both protons and neutrons make full use of the space available. Assuming that the nuclear charge is uniformly distributed over the volume of a sphere then its electrostatic potential energy can easily be shown to be $\frac{3}{5}(Ze)^2/4\pi\varepsilon_0 R$. If the discrete nature of the charge on the proton is taken into account, perhaps Z^2 should be replaced by $Z(Z-1)$. This refinement is probably an improvement but, except for small nuclei, it makes little difference in the mass

equation. In any case there are larger factors of ignorance: the charge distribution is known to vary with radius within the nucleus, and there could be a correlated motion which allows protons to avoid each other yet retains the required charge distribution. The electrostatic term is therefore introduced into the mass equation as $+\varepsilon Z^2 A^{-1/3}$, where the (unknown) constant ε is used to account for these uncertainties.

At this stage the mass equation is

$$M(A,Z) = Z(m_p + m_e) + (A-Z)m_n - \alpha A + \beta A^{2/3} + \varepsilon Z^2 A^{-1/3}.$$

Minimizing this function with respect to Z at constant A gives, for stable nuclei, the approximate result $Z = A^{1/3}(m_n - m_p - m_e)/2\varepsilon$. The important factor is $Z \propto A^{1/3}$, which is far too slow a variation with A (see Fig.1.1). $A = 32$ has $Z = 16$ stable, so the above

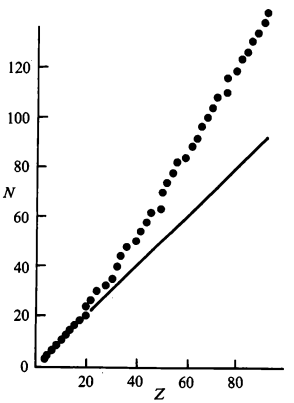

Fig.1.1. Plot of N against Z of most stable nucleus for given A. For convenience only every sixth A-value is plotted. The full line shows $N = Z$.

formula would give $Z = 32$ stable for $A = 256$, to be compared with the naturally stable $^{238}_{92}U$ having $A = 238$, $Z = 92$.

It is interesting to note from Fig.1.1 that, up to $A = 40$, the line of stability has $N = Z$. Above this value N increases more rapidly than Z, presumably because of the electrostatic term in the mass equation. It would appear that a term $\propto (N-Z)^2$, i.e. $(A-2Z)^2$ is also required, which if of positive sign would mitigate in favour of $N = Z$ for stable nuclei. One need not look far for the origin of such a term; the Pauli principle will ensure that the nucleus (in the absence of the electrostatic term) should have $N \sim Z$ in order to utilize the lowest levels available to both species. To proceed further it is necessary to invoke a model. A convenient model, which incorporates the Pauli principle, is to consider the neutrons and protons as two non-interacting degenerate Fermi gases inside the nucleus. Neglecting the electrostatic term and the n-p mass difference, which are already in the mass equation, the model instantly leads to $N = Z$ for minimum energy. Expanding the energy function from the point $N = Z$ gives a leading term $\propto (N-Z)^2/A$. Hence a symmetry term $+\gamma(A-2Z)^2/A$ is added to the mass equation.

An extra term is needed before attempting to use the mass equation to determine which nuclei are stable. In Appendix B it is shown, under approximation, that the balance between terms so far developed in predicting the heavy stable nuclei is so finely poised that the nearest nucleus (of given A) to the line of stability will be most stable. Thus, starting from an even-Z even-N stable nucleus, such that the next

stable mass is obtained by adding a neutron, then the second nucleon added to form a stable nucleus will be a proton more often than a neutron since, if (A,Z) is on the line of stability then $(A + 2, Z + 1)$ will be nearer that line than $(A + 2,Z)$. Thus, for any given A, only one stable nucleus will be expected, and of those with even A, stable nuclei will occur as often with odd Z as with even Z.

Nuclear charts show that there are 166 even-Z even-N stable nuclei but only 4 of odd Z, odd N, and these are very light: ${}^{2}_{1}H$, ${}^{6}_{3}Li$, ${}^{10}_{5}B$, ${}^{14}_{7}N$. Conditions for very light nuclei do not conform to those under which the mass equation was developed, and in addition the systems have few widely-spaced energy levels which could affect the balance between Z and N. Of the 105 odd-A stable nuclei, 50 are found to have odd Z.

In order to account for this distribution of stable nuclei, a pairing energy is postulated and is accounted for in the mass equation by a small term $\delta(A,Z)$, where δ is a function which has the value zero for all odd A and takes a negative value when both A and Z are even but a positive value when A is even and Z is odd. Its magnitude drops off as A increases, and this is to be expected since, on average, the two similar pairing nucleons will be further apart for large A than for small A.

The atomic mass equation therefore reads

$$M(Z,A) = Z(m_p + m_e) + (A-Z)m_n - \alpha A + \beta A^{2/3} + \frac{\gamma(A-2Z)^2}{A} + \frac{\varepsilon Z^2}{A^{1/3}} + \delta(A,Z) \quad . \tag{1.1}$$

This five-parameter formula has been fitted to the

available mass data (several hundred nuclei neglecting the very light ones, $A < 20$) to give values:†

Parameter	α	β	γ	ε	δ
Value (MeV)	15·6	17·2·	23·3	0·7	$33\frac{1}{2}\ A^{-\frac{3}{4}}$

The fit is shown in Fig.1.2, where it was more convenient to present the binding energy per nucleon (B/A)

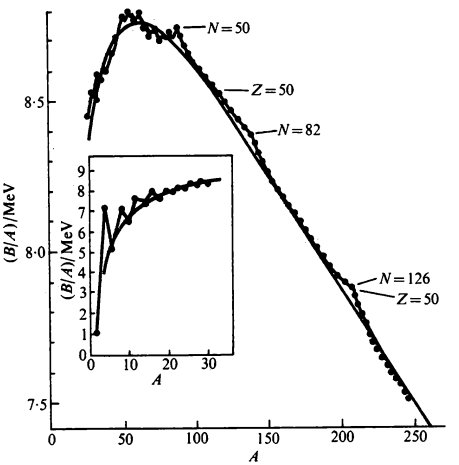

Fig.1.2. Plot showing the binding energy per nucleon as a function of A. Locations of magic numbers are indicated. The inset continues the plot and illustrates α-particle structure in light nuclei. (Based on *Wapstra* (1958). Atomic masses of nuclides, *Handb.Phys.* XXXVIII/I.

†The values of the constants vary from source to source. Usually no errors are given so the significance of the variations cannot be assessed. The fits appear to predict the masses of stable nuclei, and their neighbours on the mass diagram, to about 1 per cent in binding energy.

against A. To give an idea of the relative importance of the terms, for the stable nucleus (125,52) the contributions to B/A are + 15·6 MeV (volume), - 3·44 MeV (surface), - 0·65 MeV (symmetry), and -3·0 MeV (charge). The variation with A is shown in Fig.1.3. Notice that, on the scale shown, the pairing energy term is quite negligible; for the neighbouring even-

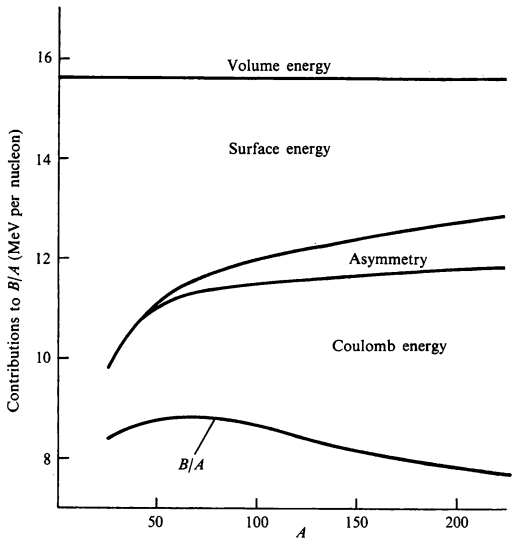

Fig.1.3. Contributions to B/A of the individual terms in the mass formula.

even nucleus to A=125, it contributes an additional 8 keV to B/A.

Stability against β-decay

β^--decay is represented by the reaction

$$(A,Z) \rightarrow (A,Z+1) + e^- + \bar{\nu}_e,$$

where ν_e and $\bar{\nu}_e$ represent electron neutrino and election antineutrino (see Chapter 4). This reaction is energetically possible if

$$m(A,Z) > m(A,Z+1) + m_e \qquad (1.2)$$

since $m_\nu \sim 0$. But

$$m(A,Z) = M(A,Z) - Z\, m_e$$

leading to

$$M(A,Z) > M(A,Z+1) \; . \qquad (1.2a)$$

β^+-decay is represented by the reaction

$$(A,Z) \rightarrow (A,Z-1) + e^+ + \nu_e$$

and is energetically possible if

$$m(A,Z) > m(A,Z-1) + m_e \qquad (1.3)$$

leading to

$$M(A,Z) > M(A,Z-1) + 2m_e . \qquad (1.3a)$$

For these two types of decay it seems that the nuclear masses produce a more attractive statement of condition. However, there is another type of decay - an alternative, competing, mode to β^+-emission - namely, electron capture. Nuclei are not required to be stable in isolation, but in their natural habitat when surrounded by atomic electrons. Any one of these

electrons can be captured, though if energetically possible K-capture is likely to be dominant, leaving the atom strongly excited. However, in discussing the limit of stability the possibility of capture of the outermost electron, leaving the atom in its lowest state, must be considered, since energetically it is most favoured.

This reaction is represented by

$$(A,Z) + e^- \rightarrow (A,Z-1) + \nu_e$$

and is energetically possible if

$$m(A,Z) + m_e > m(A,Z-1), \tag{1.4}$$

leading to

$$M(A,Z) > M(A,Z-1). \tag{1.4a}$$

These equations indicate that electron capture can proceed when β^+-emission is impossible and also that the atomic mass scale is again favoured: the condition for stability against the β-decay process is that the *atom* (A,Z) should have least mass.

For the heavier nuclei this can be expressed, approximately, as $(\partial M/\partial Z)_A = 0$, which is more convenient to use than the two inequalities. The corresponding condition on the nuclear mass scale $(\partial m/\partial Z)_A = -m_e$, is probably not so convenient to remember nor indeed is the differential condition on the binding energy, $(\partial B/\partial Z)_A = m_p + m_e - m_n$.

From these equations, two nuclei (A,Z) and $(A,Z-1)$ can both be stable only if they have exactly

the same atomic mass. This statement is subject to approximations concerning the neutrino mass and the electron binding energies. Taking these into account might produce a band of stability a few eV wide. The chance that such a pair of nuclei should have equal atomic masses to within a few eV is remote, so it is concluded that, in the few such examples naturally occurring, one of the pair is unstable but has a very long lifetime, e.g. ^{87}Rb and ^{87}Sr - (87,37) and (87,38) - of which the former has a lifetime of 5×10^{10} years, emitting β^--particles with maximum energy $\sim$ 274 keV.

Effect of the pairing term on stability

When A is constant and odd the mass values for different Z will define discrete points lying on a parabola. For A constant but even, it is necessary to define two parabolae displaced from each other by $2\delta(A)$, which is roughly 2·5 MeV at A = 81, falling to $\sim$ 1 MeV at A = 256. The odd-Z mass points will fall on the upper parabola and the even-Z mass points on the lower. Fig.1.4 gives examples of the parabolae for odd A and even A.

The displacement of the two parabolae for even A is sufficient to ensure, even when the lowest odd-Z mass point is at the minimum of its parabola, that both its neighbours (which will lie on the lower parabola) have lower mass. This can be checked for the two masses given above by evaluating the second derivative of the mass formula without the δ term; the mass difference between this odd nucleus, assumed to be at the minimum of the parabola, and its two even neighbours will then by $\frac{1}{2}(\partial^2 M/\partial Z^2)_A$. At A = 81 this is $\sim$ 1·5 MeV and at A = 256, $\sim$ 0·6 MeV, so that the

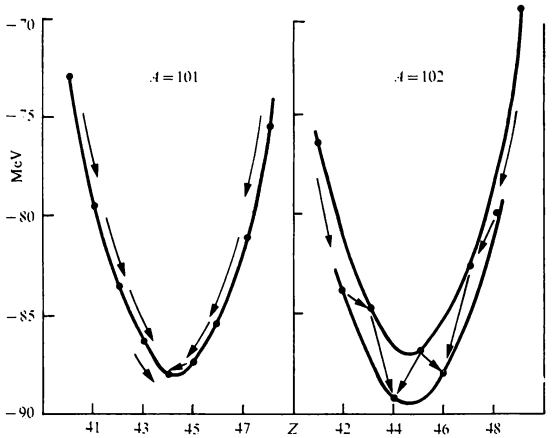

Fig.1.4. Mass parabolae for odd A and even A. The mass excess (in MeV) is plotted against Z for A = 101 and 102.

pairing term is amply large enough for its purpose.

These considerations lead to the conclusions that, for odd A, only one nuclide is expected to be stable against β-decay, but for even A two apparently stable nuclides is likely to be a common occurrence (and is) and even three is a remote possibility (one or two examples are found). The term 'apparently stable' has been used because one such stable even-even nucleus should be able to decay to its less massive even-even neighbour by a double β-decay process, which as yet has not been observed.

Stability against nucleon or α-particle emission

From Fig.1.2 it is seen that for $A > 20$ the binding per particle for nuclei near the minima of the mass

parabolae (i.e. along the 'valley of stability' on a three-dimensional M,A,Z plot) lies between 6 MeV and 8 MeV. This large value ensures that even the most massive naturally occurring nuclei ($A \sim 240$) are stable against emission of proton or neutron.

The α-particle, on the other hand, is itself bound by $\sim$ 28 MeV, so that instability to α-emission is much more probable. The condition for instability is

$$M(A,Z) > M(A-4,\ Z-2) + M(4,2)$$

or

$$B(A,Z) < B(A-4,\ Z-2) + B(4,2)\ .$$

However, Fig.1.2 presents B/A as a function of A; if we denote this by $\phi(A)$ - along the line of stability Z is a function of A, and so $B(A,Z)$ is a function of A only - then

$$A\ \phi(A) < (A-4)\ \phi(A-4) + 4\ \phi(4)\ .$$

But

$$\phi(A-4) \approx \phi(A) - 4\ d\phi/dA$$

giving, after dividing by 4,

$$\phi(A) < \phi(4) - (A-4)\ d\phi/dA\ .$$

From the graph $d\phi/dA \sim 0{\cdot}75$ MeV/100 and is roughly constant well above the maximum of the curve; and

$\phi(4) \sim 7$ MeV.

Thus the condition for α-instability is that $\phi(A)$ is less than $\sim 7{\cdot}8$ MeV at $A = 100$, $\sim 8{\cdot}1$ MeV at $A = 150$, and $\sim 8{\cdot}5$ MeV at $A = 200$. α-decay is therefore possible for $A > 150$ (though shell effects have an influence; see Chapter 3). Because of the mechanism of the process (see Chapter 4) it becomes a significant mode of decay only above $A \sim 210$.

ANGULAR MOMENTUM (SPIN) AND STATISTICS

Quantum mechanically, as classically, an isolated system has a well-defined angular momentum. It is therefore a characteristic property of the nuclear ground state - and indeed any other well-defined state. As such it is loosely referred to as spin of the nucleus, even though it is a combination of the intrinsic spins and orbital motions of the individual nucleons. The reason for this is probably historical. Nuclear spins of stable and long-lived nuclei were determined from their magnetic interactions with the atomic electrons or with externally applied fields. For such interactions the spin vector is an intrinsic property of the nucleus just as it is, at present, for an elementary particle.

Since the intrinsic spin of a nucleon is $\frac{1}{2}$, whilst all orbital angular momentum vectors are characterized by integer quantum numbers, it follows that the sum of A such spins and orbital terms will lead to an angular-momentum (or spin) quantum number J which will be an integer when A is even but (integer + $\frac{1}{2}$) when A is odd. Examination of a table of nuclear spins confirms this and also reveals a striking fact: all even-Z even-N nuclei have ground-state spin 0, without exception.

Since a pairing energy term, favouring the pairing of like nucleons, has already been introduced into the mass equation, it seems reasonable to suggest that such pairs individually couple to zero angular momentum. This point will be taken up in Chapter 3.

The Pauli principle requires that quantum-mechanical systems have wavefunctions which are antisymmetric to the exchange of like fundamental particles. If the system under consideration is atomic (for definiteness, consider a molecule made up of two like atoms) then the electronic wavefunction should have a definite symmetry on interchange of two nuclei as a whole, whilst the nuclear wavefunction should be antisymmetric on interchange of two protons (or neutrons), either within a nucleus or between two nuclei. If the like nuclei have mass number A then their interchange is equivalent to A interchanges of nucleons. Thus the electronic wavefunction will be antisymmetric for odd A and symmetric for even A. The symmetry reflects in the statistical behaviour of an aggregate of such atomic systems. Antisymmetric states conform to the Fermi-Dirac distribution, whilst symmetric states conform to the Bose-Einstein distribution (Martin 1975; Pauling and Wilson 1935); for a diatomic molecule of similar atoms, the intensities of lines in a rotational band spectrum reflect the statistics obeyed and so determine whether the number of nuclear particle components is odd or even (see Burcham 1963). This may sound trivial now, but before the discovery of the neutron it led to rejection of the hypothesis that nuclei were composed of protons and electrons. In particular, for ^{14}N (14,7) the N_2 molecule obeys

Bose-Einstein statistics, although on that theory ^{14}N was thought to consist of 14 protons and 7 electrons - an odd total of fermions.

The symmetry rules impinge on nuclear reactions in a somewhat different way. Consider the system (α+α); its wavefunction must be symmetric under the interchange of the two α-particles. If these particles can come together to form a state in ^{8}Be then this state must reflect that symmetry. But ^{8}Be is more complex than a (2α) structure (e.g. it can also be formed from ^{7}Li + p) and each state can be looked upon as a conglomerate of such structures. However, of all the states in ^{8}Be which exist only those with a certain symmetry can decay into α+α. These states must have J= 0, 2, 4, etc., as can be shown by a consideration of the wavefunctions for α+α.

PARITY

Parity is another aspect of symmetry in quantum mechanics, but one which has no direct correspondence in the classical limit. If the interaction potential of a system has the symmetry property $V(x,y,z) = V(-x,-y,-z) = PV(x,y,z)$, where the operator P has been introduced and defined, and the Schrödinger wave equation has a solution $\psi(x,y,z)$, then $P\psi(x,y,z)$ is also a solution. For a non-degenerate case, the second solution must be a numerical factor times the first, i.e. $P\psi(x,y,z) = \alpha\psi(x,y,z)$. But it is obvious that $P^2 = 1$, corresponding to two inversions, so that $\alpha = \pm 1$. Thus a non-degenerate solution corresponding to this type of potential has either even parity ($P = +1$) or odd parity ($P = -1$).

For a single particle in a central potential there

is a one-to-one correspondence between the orbital angular momentum and the parity, which follows from the angular solution of the problem:

$$\text{even } l \equiv \text{even parity}, \quad \text{odd } l \equiv \text{odd parity}.$$

For a multi-particle assembly like the nucleus, short of a complete solution to the problem, the parity can only be deduced in terms of a model in which the parities of the individual particles can be specified, as in the shell model (Chapter 3). From the fact that the complete wavefunction can be expanded in terms which are themselves products of the wavefunctions of the individual particles, it follows that the overall parity is a product of the individual parities if + is assigned to even parity and - to odd parity.

As a constant of the motion, parity plays an important part in nuclear reactions. If in the reaction $X + a \rightarrow W^* \rightarrow Y + b$, where W^* represents a state in the compound nucleus (Chapter 6), the spins and parities of all five particles are known, then conservation of angular momentum and parity severely restrict the orbital angular momentum taken in by a and taken out by b, thus affecting both the angular distribution and the magnitude of the cross-section. As an example assume in the above that X has (spin, parity) = (0^+), a is a proton $(\frac{1}{2}^+)$, b is a neutron $(\frac{1}{2}^+)$, Y is 2^+, the compound nucleus W^* is $\frac{3}{2}^-$. Then, conserving angular momentum only gives $l_a = 1$ or 2, $l_b = 0,1,2,3$, or 4, but parity conservation will reduce these possibilities to $l_a = 1$, $l_b = 1$ or 3. For spinless particle and projectile, the accessible states in the compound nucleus are severely restricted, e.g. for $^{12}C + \alpha \rightarrow {}^{16}O^*$

then the excited states must be 0^+, 1^-, 2^+, 3^-, etc.

ELECTRIC AND MAGNETIC MOMENTS

Electric moments

If a charge is distributed over a finite region of space, surrounding a chosen origin, through which an electric field derivable from a potential $U(x,y,z)$ has been imposed from outside, then the interaction energy can be expressed in a Taylor series thus,

$$E = \int U(x,y,z)\rho(x,y,z)d\tau = U(0)\int \rho d\tau +$$

$$+ \left(\frac{\partial U}{\partial x}\right)_0 \int x\rho d\tau + \left(\frac{\partial U}{\partial y}\right)_0 \int y\rho d\tau + \left(\frac{\partial U}{\partial z}\right)_0 \int z\rho d\tau +$$

$$+ \tfrac{1}{2}\left(\frac{\partial^2 U}{\partial x^2}\right)_0 \int x^2\rho d\tau + \tfrac{1}{2}\left(\frac{\partial^2 U}{\partial y^2}\right)_0 \int y^2\rho d\tau + \tfrac{1}{2}\left(\frac{\partial^2 U}{\partial z^2}\right)_0 \int z^2\rho d\tau +$$

$$+ \left(\frac{\partial^2 U}{\partial x \partial y}\right)_0 \int xy\rho d\tau + \left(\frac{\partial^2 U}{\partial y \partial z}\right)_0 \int yz\rho d\tau + \left(\frac{\partial^2 U}{\partial z \partial x}\right)_0 \int zx\rho d\tau +$$

+ higher-order terms.

The first term gives the interaction energy with the field of a point charge at the origin equal to the total charge - the monopole term. The next three terms represent the scalar product of a vector dipole - the dipole moment of the charge distribution - with the vector field at the origin; whilst the next six terms represent the scalar product of two tensors, one the generalized derivative of the vector field and the other the quadrupole moment of the charge distribution. These classical definitions hold for a quantum-

mechanical system if we interpret ρ as $e \sum_i \psi_i^*(r)\psi_i(r)$, where the summation i is taken over the charged particles of the system (protons in the nucleus).

The fact that a nucleus has a well-defined parity makes $\rho(\underline{r}) = \rho(-\underline{r})$. Thus every odd moment - dipole, octupole, etc. - will vanish identically. The simplest deformation from spherical results in an ellipsoid of revolution, which, if the z-axis is the symmetry axis, produces a quadrupole-moment tensor which can be specified by a single parameter Q_0 given by $eQ_0 = \int(3z^2 - r^2)\rho d\tau$ - a definition which conveniently has $Q_0 = 0$ for spherical symmetry. A positive value for Q_0 corresponds to a prolate shape for the nucleus, i.e. elongated along the symmetry axis; whilst negative Q_0 defines an oblate nucleus, i.e. flattened along the symmetry axis. As defined the dimensions of Q are those of an area for which the convenient nuclear unit is the barn (b)(1 barn = 10^{-24}cm^2 = 10^{-28}m^2). Carrying out the integral for ρ = constant within the ellipsoid gives $Q_0 = \frac{2}{5} Z(a^2-b^2) \approx \frac{4}{5} ZR^2\left(\frac{\Delta R}{R}\right)$, where a, b are the major and minor semi-axes, R the average nuclear radius, and $\Delta R = a-b$.

Obviously a state $J = 0$ has spherical symmetry and so has zero observable quadrupole moment; less obviously, a state $J = \frac{1}{2}$ also has spherical spatial symmetry and zero observable quadrupole moment. Table 1.1 illustrates these points and also indicates that some nuclei can be strongly deformed with $\Delta R/R \sim 30$ per cent.

Magnetic dipole moment

A similar, but more complicated, expansion of the

TABLE 1.1

Nucleus	B/MeV †	Atomic binding keV	GROUND STATE J^π	μ/nm	Q/b	$\Delta R/R$ ††
n	-		$\frac{1}{2}^+$	-1·9135	-	
$^{1}_{1}$H	-	0·014	$\frac{1}{2}^+$	+2·79275	-	
$^{2}_{1}$H	2·225	0·014	1^+	+0·85735	+0·00282	?
$^{4}_{2}$He	28·29	0·079	0^+	0	0	0
$^{7}_{3}$Li	39·24	0·190	$\frac{3}{2}^+$	+3·2564	-0·04	?
$^{16}_{8}$O	127·6	2·0	0^+	0	0	0
$^{35}_{17}$Cl	298·2	12	$\frac{3}{2}^+$	+0·8218	-0·08	0·1
$^{57}_{26}$Fe	499·9	34	$\frac{1}{2}^-$	+0·0905	0	0
$^{121}_{51}$Sb	1026	180	$\frac{5}{2}^+$	+3·342	-0·53	-0·12
$^{176}_{71}$Lu	1418	370	$\frac{7}{2}^-$	+3·180	+8·0	+0·50
$^{181}_{73}$Ta	1452	390	$\frac{7}{2}^+$	+2·35	+3·9	+0·35
$^{209}_{83}$Bi	1640	550	$\frac{9}{2}^-$	+4·080	-0·34	-0·02
$^{238}_{92}$U	1783	690	$\frac{7}{2}^-$	±0·35	±4·1	±0·25

Table 1.1 continued

Footnotes: † B is given only to four figures; it is however known to the nearest keV for the light elements and somewhat less precisely for the heavy.

†† The $\Delta R/R$ have been computed using $R = 1{\cdot}1A^{1/3}$ fm. A relationship $Q/Q_0 = J(2J-1)/(J+1)(2J+3)$ is used to connect the observed quadrupole moment Q with the intrinsic quadrupole moment Q_0.

charge distribution and its interaction with an external magnetic field, taking into account a possible circulation of the charge within the distribution will give a leading term dipole in nature. If in transition to the particle system the intrinsic magnetic moments of the individual particles are taken into account, this dipole term will have two parts: (1) from the circulating currents of the proton distribution which will be proportional to some suitable summation of the orbital angular momenta l_i and (2) from a suitable summation of the intrinsic magnetic moments of both protons and neutrons which will be vectors parallel to the spins s_k. The phrase 'some suitable summation' is used because the resultant magnetic moment of a number of particles depends on how the particles are coupled together; recourse to a model is necessary, as will be apparent in Chapter 3. At this stage it should be noted that the magnetic dipole moment is a vector, and it is a fundamental result in quantum mechanics that all vector constants of a system are scale factors of one vector, which, for an isolated

system, will be the total angular momentum J. Thus vectorially $\underline{\mu} = g\underline{J}$, where g is the suitable scale factor. It is convenient to represent the dipole moment by a scalar, which in the classical limit would represent the magnitude $|\mu|$. Two possibilities exist: $g\{J(J+1)\}^{\frac{1}{2}}$ and $g(m)_{max} = gJ$. The latter has been adopted.

For the case of a spinless particle of mass m and charge e, J = L and g is the gyromagnetic ratio $e/2m$ - a relationship which can readily be verified for a circular orbit, using Ampere's theorem connecting a circulating current with a magnetic shell. This relationship leads to a natural unit $e\hbar/2m$ for the dipole moment. In atoms the mass is that of the electron, defining the Bohr magneton (μ_B); in nuclei it is the mass of the proton defining the nuclear magneton (μ_N), which has the value $5{\cdot}051 \times 10^{-27}$ J T^{-1}. The intrinsic magnetic dipole moments of the proton and neutron are respectively $+2{\cdot}79275\ \mu_N$ and $-1{\cdot}9135\ \mu_N$. A few nuclear magnetic moments are given in Table 1.1.

PROBLEMS[†]

1.1. Assuming that the nuclear size parameters are as on p. 12, calculate the quantum number n for which the circular Bohr orbit of K^- around $^{208}_{82}Pb$ lies in nuclear matter at a density of 10 per cent of the central nuclear density. The mass of the K^- is 494 MeV/c^2.

1.2. By assuming that the neutrons and protons in the

[†] * indicates that the problem is taken from, or based on, an Oxford Finals question.

nucleus behave as independent Fermi gases, and by neglecting the Coulomb interaction, derive the form of the symmetry term, $\propto (N-Z)^2$, which appears in the mass formula.

1.3. Verify the statement that the pairing term in the mass equation is sufficiently large to ensure no odd-odd nucleus (above $A=40$) is less massive than both its even-even neighbours of the same mass number.

1.4*. Calculate values (in MeV/c^2) for γ and ϵ in the mass equation from the following facts: $^{35}_{18}$A emits positrons with a maximum energy of 4·95 MeV; $^{135}_{56}$Ba is the stable isobar of mass number 135.

1.5*. The maximum energy of the positrons from the decay of $^{13}_{7}$N is 1·24 MeV and there is no subsequent γ-radiation. Deduce a value for the radius of nuclei of mass 13. (The n-p mass difference is 1·29 MeV/c^2.)

1.6. Show that the angular wavefunction $\sin\theta \exp(i\phi)$, corresponding to $l=1$, $m_l=1$, has odd parity, whilst $\sin^2\theta \exp(i2\phi)$ and $\sin\theta \cos\theta \exp(i\phi)$, corresponding to $l=2$, $m_l=2$ and 1, have even parity.

1.7. Deduce the gyromagnetic ratio for the case of a charged particle moving in a circular orbit. Hence derive the natural units for the magnetic moments of electron and proton orbits.

2. Nuclear forces

INTRODUCTION

From the last chapter a picture of the nucleus has emerged which is quite different from that of an atom. In the latter, the small massive central nucleus interacts more strongly ($\sim Z$ times) with any electron than that electron does with any other electron. For the nucleus there is no well-defined centre; the interaction of one nucleon with its neighbour at any instant of time is probably stronger than its interaction with the nucleus as a whole - this latter being a time-averaged effect of all the individual interactions. Thus the problem is a many-body one, without the obvious approximations of the atomic case; in the next chapter models will be illustrated which simplify the many-body aspects. The shell model has been taken over from the atom, but initially with reluctance, since its basic concepts seem to clash with ideas of the nucleus.

In this chapter, the basic nucleon-nucleon interaction is examined in order to answer the questions:

(1) Assuming that the strong interaction between nucleons can be represented by a potential function, what is its basic shape and does it depend on the type of nucleon?

(2) Arguing backwards from a nucleus towards a basic nucleon-nucleon interaction, does this interaction agree with that deduced from basic nucleon-nucleon scattering experiments?

(3) Knowing that the answer to (2) is indefinite, is there a need to introduce new forces in the nucleus which are not present in the basic two-body interaction?

The answer to the third question can be given now. Theorists are not happy with the idea of many-body forces, although there appears to be no very good reason why they should be small compared with the two-body interaction. The subject has not yet reached such a stage of refinement as to need a small 'topping up' with three-body forces. If they are to be introduced at all, it may as well be done in bulk - in which case what of four-body forces, etc? The hope appears to be that averaging many-body forces in a large nucleus produces much the same effect as averaging two-body forces. Note that, in answering this question, addition of a third nucleon will always modify the interaction between two nucleons because of the Pauli principle - but this effect is always allowed for in any attempt to solve the problem of larger nuclei and is not to be looked upon as a new three-body interaction in the sense referred to here.

INFERENCES FROM THE DATA PRESENTED

In developing the nuclear properties of the previous chapter, a number of facts have emerged which have a direct bearing on the forces which hold the nucleus together. Taken roughly in the order in which they were presented they are:

(1) nuclear sizes are of the order of a few fm;

(2) nuclear density, at least near the centre, is the same for all nuclei;

(3) protons and neutrons have similar density distributions inside the nucleus;

(4) scattering of nuclear projectiles gives the Rutherford cross-section up to a certain angle with a rapid fall-off above this angle;

(5) the basis for a successful analysis of electrostatic energy differences between mirror pairs is that nuclear forces are 'charge symmetric', i.e. pp $\equiv$ nn;

(6) the nuclear density falls off over a range of $\sim$ 1 fm at the periphery;

(7) the leading terms in the binding energy, $\alpha A - \beta A^{2/3}$, were arrived at by analogy with a liquid drop;

(8) there is a 'symmetry term' in the mass equation;

(9) the existence of a pairing term, and the coupling to zero angular momentum of every ground state of even A indicate a pairing force;

(10) the α-particle is almost as tightly bound as a massive nucleus;

(11) some heavy nuclei have large quadrupole moments;

(12) even the deuteron has a quadrupole moment;

(13) the proton and neutron have 'anomalous' magnetic moments, i.e. different from those for structureless Dirac particles;

(14) the magnetic moment of the deuteron is close to, but not equal to, the sum of the proton and neutron magnetic moments.

RANGE AND SATURATION

The terms 'long-range' and 'short-range' have both a general and a specific meaning. In the general sense long and short are relative to the size of the system; van der Waals forces of cohesion of a liquid drop are short-range in the sense that the binding of a given molecule is dominated by the contributions of a few fairly close neighbours rather than by the overwhelmingly greater number of distant molecules - these forces $\propto 1/r^7$ (or so) in energy-dependence. Likewise, in nuclei, forces are short-range if the contribution of most nucleons to the binding of a given one is less than the contribution from its near neighbours. In the specific sense, forces are said to have a range if the potential function contains an exponential, e.g. Yukawa forces having a potential function $U \propto (1/r)\exp(-kr)$ are said to have a range $1/k$. Thus it will be seen that nuclear forces probably are short-range in both the general and specific senses, though arguments on saturation are concerned really with the general sense.

The nuclear force field extends beyond the nuclear surface by a distance of the order of magnitude of the range of nuclear forces, so (1) and (4) (of the previous section) taken in conjunction can be used to define this range. The estimate obtained is confirmed by (6) since we would, in a simple way, expect the fall-off of nuclear density to occur over the same range.

(2) and (7) in conjunction with the liquid-drop model from which the mass terms were derived do much to define nuclear forces. The molecular force field in a liquid drop has two main components: the attrac-

tive short-range force, which can be oversimplified by describing it as an induced-dipole - induced-dipole interaction for which the potential energy $\propto r^{-6}$; and the repulsive, even shorter-range force which occurs when the electron clouds overlap. By analogy, a similar state of affairs may be expected to occur in nuclei; the short-range attraction would require a range of 1 - 2 fm, whilst the shorter-range repulsion might have a range of less than 0·5 fm.

Oddly enough, in the historical development of the subject, the concept of a repulsion at short distance was not readily acceptable. Nucleon-nucleon scattering had not yet indicated the need for such a force, which appeared to require a complex structure for what was hoped to be a simple fundamental particle. Instead, recourse was made to the exchange phenomenon (see Appendix C) to provide forces with suitable properties. Such forces were described as: Wigner (W) or ordinary (no exchange of quantum numbers); Majorana (M) or space-exchange (corresponding to exchange of charge and spin); Heisenberg (H) or charge-exchange (but not spin); and Bartlett (B) or spin-exchange (but not charge). All these forces can arise from pion-exchange. From analogy with electron-exchange binding of molecules, a nucleon should be surrounded by a pion cloud. That this is so can be inferred from the magnetic moments of proton and neutron, which for simple Dirac particles should be 1 μ_N and 0 μ_N; but the pion has a much smaller mass than proton and neutron so the cloud can, and does, make a large contribution to their moments.

Exchange forces will depend upon whether the property being exchanged is symmetric or anti-symmetric

in the wavefunction representing the relative motion of the two particles. As an example the spin-exchange operator (see Appendix C), $P^{\sigma} = \frac{1}{2}(1+4\underline{s}_1 \cdot \underline{s}_2)$, has the value +1 for $S = 1$ and -1 for $S = 0$, where $\underline{S} = \underline{s}_1 + \underline{s}_2$, so the spin-exchange force can be made attractive for $S = 1$ and repulsive for $S = 0$, or vice versa.

How do exchange forces lead to saturation? In a heavy nucleus A, the number of nucleon-nucleon interactions is $\frac{1}{2}A(A-1)$. If the forces are ordinary then all these are increasing the binding of the nucleus, and a complete calculation leads to all nuclei having roughly the same size with a binding energy proportional to A^2 rather than A. However, the operation of the Pauli principle ensures that all nucleon pairs cannot have the same spin symmetry or charge symmetry or spatial symmetry, so that exchange forces will as often be repulsive as attractive. Their overall effect need not be zero, since the strength of the force will depend upon a spatial overlap integral which of course is largest when the spatial wavefunctions of the two particles are the same. The exclusion principle, in its restricted statement, limits the number of particles with the same spatial wavefunction to 4 - two protons (spin up, spin down) and two neutrons. Thus Majorana (spatial) exchange forces might be expected to produce strong interactions within groups of four particles and weak effects on averaging over pairs not in the same group. The emergence of four particles is an attractive result because (10) emphasizes the fact that the α-particle is about as strongly bound as any heavy nucleus.

Much effort has gone into producing a recipe for

nuclear forces using these four types - but each recipe, whilst successful for explaining the particular fact on which the 'cook' was concentrating, has had its shortcomings, and the fact of a repulsive core (see p. 54) has been finally accepted with relief. This does not mean that exchange forces are unnecessary; it is difficult to see how ordinary forces with repulsion could lead to a saturated α-particle. Indeed, from the range of the repulsive force as required from nucleon-nucleon scattering, it is possible to attach an effective size to each nucleon ($r_{\text{eff}} \sim \frac{1}{2} \times$ range, since two nucleons cannot approach closer than this range). The effective volume of the nucleons in a heavy nucleus is then found to be $\sim$ 1 per cent of the nuclear volume. At its face value this indicates that saturation is largely accounted for by exchange forces, though no doubt the repulsive core also plays an important role.

ISOSPIN

In (5) (see p. 38) the point is made that forces are 'charge symmetric', by which it means that the (p,p) and (n,n) interactions are the same, apart from the electromagnetic contribution, but the (p,n) could be different. In the analysis of 'mirror paris' the numbers of (p,p) and (n,n) interactions are interchanged but the number of the nuclear-force binding does not specify the last interaction.

A further step to 'charge independence' is made by assuming that all three interactions are the same; this can be checked by analysing nucleon-nucleon scattering or by a more detailed examination of nu-

clear properties. Charge independence is expressed mathematically using the isospin formalism, which was developed as a convenience before it acquired physical significance. Rather than referring to two distinct types of particle within a nucleus, theorists preferred to employ a nucleon to which an additional two-valued quantum number was assigned and given the value $+\frac{1}{2}$ for the neutron and $-\frac{1}{2}$ for the proton. The reason for this choice is that all two-valued quantum numbers have a similar mathematical behaviour, so the values reflect the similarity with $S_z = \pm\frac{1}{2}$ for the spin quantum number.

The name 'isospin' is a shortening of 'isobaric spin', indicating that the two components have the same mass and emphasizing the similarity with spin S. By analogy, there exists a vector $\underline{t}$ and a component t_z of isospin, but the space in which this vector exists is not the same as the space in which ordinary spin exists. Thus, in this space, $t_z = +\frac{1}{2}$ represents a neutron and $t_z = -\frac{1}{2}$ a proton.† Pursuing the analogy, a vector sum $\underline{T} = \underline{t}_1 + \underline{t}_2$, corresponding to $\underline{S} = \underline{s}_1 + \underline{s}_2$, can be made for two (and more) nucleons. A nucleon pair can have $T = 1$ giving a function symmetric on exchange of nucleons, or $T = 0$, antisymmetric. If the Pauli principle is extended to make the overall wavefunction antisymmetric with the isospin component included then it can be seen that the same states in

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†It is unfortunate that particle physicists have later chosen precisely the opposite convention - in principle it does not matter, since the direction up or down in an unspecified space is hardly relevant, but it is confusing.

the two nucleon configuration are obtained whether or not this formalism is used; since (p,p) and (n,n) are symmetric in isospin their wavefunctions must be antisymmetric in (space) (spin), but the (n,p) combination can be symmetric or antisymmetric in isospin (T = 1, T_z = 0 represents (p,n) and is symmetric; T = 0, T_z = 0 also represents (p,n) and is antisymmetric) so the (space) (spin) wavefunctions can also be either.

Thus far, the formalism is merely mathematical, but charge independence can readily be fed into the scheme by asserting that nuclear forces depend only on T and not on T_z. This achieves its aim because the selection of T determines the type of (space) (spin) wavefunction and so ensures that we compare like with like when testing the similarity of (p,p),(p,n), and (n,n) interactions. The equality of strong interactions in mirror pairs of nuclei follows from this statement, at least for $|N-Z|$ = 1, because these mirror pairs have $T_z = \pm \frac{1}{2}$ and, moreover, have the same value for T, also $\frac{1}{2}$. Extending to the case $|N-Z|$ = 2; ^{14}C and ^{14}O, corresponding to interchange of neutrons and protons, have the same value for T(= 1), but $T_z = \pm 1$. But charge independence further requires that there should exist a state in ^{14}N having T = 1, T_z = 0 with similar strong-interaction energy as the ground states of ^{14}C and ^{14}O. The same should apply to excited states in ^{14}C and ^{14}O. The state of affairs is shown in Fig.2.1, where, to emphasize the point, the nuclei have been moved relative to each other through energies to allow for the electrostatic term and the p-n mass difference. An interesting feature is that the ground state of ^{14}N has T = 0; there is a general tendency for states of lower T to lie lower which re-

sults in the symmetry term in the mass equation (point (8) of p. 38). The method used to derive it implied charge independence since it treated neutrons and protons in a similar way, which again is implied in point (3).

Thus another quantum number has been introduced into the description of a nuclear state but, as the shift of energy in Fig.2.1 has perhaps indicated, it

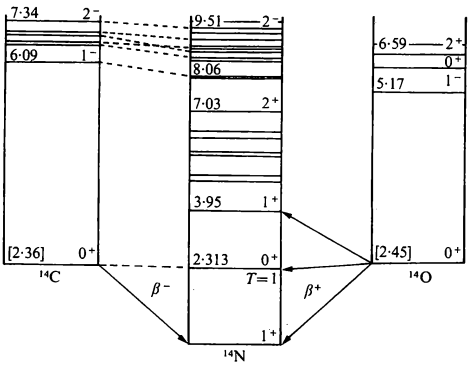

Fig.2.1. Comparison of energy levels in ^{14}C, ^{14}N, and ^{14}O. Allowance has been made for the Coulomb energy and p-n mass difference. Note that ^{14}C and ^{14}O have $T = 1$ states only, whereas ^{14}N has $T = 1$ states and a greater number of $T = 0$ states. The dotted lines guide the eye to $T = 1$ states in ^{14}N. (Based on AJZENBERG - SELOVE (1970). *Nucl.phys.* A152, 1.)

is only an approximate one, since the electromagnetic term, whilst splitting the isospin sub-states by a large energy (several MeV), is also of a form which mixes states with different T to some (small) extent.

This quantum number plays a significant role in nuclear reactions; e.g. the reaction $^{16}O + d \rightarrow {}^{14}N + \alpha$ leaves ^{14}N in excited states which are characterized by $T = 0$, since ^{16}O (ground state), d, and α all have $T = 0$. Since the state at 2.31 MeV belongs to the $T = 1$ triplet which includes the ground states of ^{14}C and ^{14}O, it can be populated in this reaction only by way of the small impurity in the wavefunction of the $T = 0$ component. This selectivity in (d,α) and similar reactions had been noticed experimentally and led to the conclusion that T was indeed quite a good quantum number.

Finally, to return to the mass equation, the pairing term (point (9)) appears to refute charge independence in favour of charge symmetry, since the term increases the binding for (p,p) and (n,n) pairs but decreases it for (p,n). However, we are failing to compare like with like; the proton pair, or neutron pair, have similar quantum numbers - this will be more obvious after reading about the shell model in the next chapter - whilst in a large nucleus the (p,n) pair are in dissimilar orbits. However, in the very light nuclei when the (p,p),(n,n), and (p,n) interactions all refer to nucleons in the same orbits (shells), it is interesting to note that the odd-odd nucleus is stable on four occasions. Thus the pairing term can be shown to be consistent with charge independence.

THE DEUTERON AND THE VIRTUAL DEUTERON

Since the deuteron presents a simple two-body problem, its study was hoped to reveal the properties of the nucleon-nucleon interaction. In hindsight it has proved to be somewhat of a disappointment; the solution

for any simple short-range potential function which might approximate to the nucleon-nucleon interaction, e.g. the square well having potential - V_0 from $r = 0$ to b and zero for $r > b$ (see Fig.2.2) indicates that,

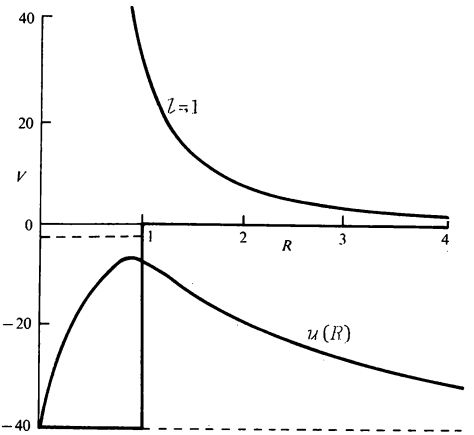

Fig.2.2. Typical square well for the deuteron, depth 40 MeV, size 1·6 fm (the scale factor for the abscissa). Also shown is the centrifugal barrier for $l = 1$, and $u(r) = rF(r)$, where $F(r)$ is the radial wavefunction of the ground state of the deuteron.

in the loosely bound deuteron, the nucleons are for most of the time outside the effective range of the force - farther apart than b in the square well quoted above. To pursue this example it turns out that the wavefunction can be very similar for a large range of variation of the square-well parameters, provided that such variation keeps $V_0 b^2$ approximately constant. The value of the constant is $\sim 10^{-24}$MeV cm^2 i.e. 1 MeV b

corresponding to something like $V_0 \sim 50$ MeV, $b \sim 1{\cdot}4$ fm This condition follows from that fact that $Kb \sim \frac{1}{2}\pi$, where K is the wavenumber inside the potential, for successful matching of the internal and external wavefunctions to produce a stationary state. For a more realistic potential without discontinuities, a similar condition can be deduced in the form of a phase integral in the region of positive kinetic energy. It need hardly be added that the analysis of the deuteron sheds no light on the existence of a repulsive core. Also, because of the fact that the deuteron size stems mainly from that part of the wavefunction corresponding to negative kinetic energy and having a (slow) exponential fall-off, feeding into the problem the (mean) size from electron scattering does not serve to pin down the force range with any precision.

An interesting result arises, however. The lowest state is undoubtedly an S-state. That this state is bound only by 2·2 MeV, in a well of depth $\sim$ 50 MeV, makes no other bound state possible; the next S-state with a node in the wavefunction requires a wavenumber of approximately 3 times that of the present state, which is clearly impossible, and the lowest $l = 1$ state is made unbound by the 'centrifugal potential'. Classically, a circular rotor with angular momentum $m\omega r^2$ has kinetic energy = $\frac{1}{2}m\omega^2 r^2$, which becomes $l(l+1)\hbar^2/2mr^2$ on quantizing the angular momentum to $\{l(l+1)\}^{\frac{1}{2}}\hbar$. A term of this form automatically arises in the radial Schrodinger equation - it is shown in Fig.2.2 for $l = 1$. Numerically, to bind a state having $l = 1$ would require a well-depth of about 200 MeV for the same range. It is highly unlikely that a mixture of exchange forces could be produced capable of such

large variation with l of the potential.

However, it is known from low-energy neutron scattering by protons that there exists a virtual state in the deuteron unbound by $\sim$ 30 KeV. The detailed analysis of scattering is beyond the scope of this discussion. Suffice it to say that from measurements of the differential cross-section it is possible to deduce the presence of resonances (or virtual states) and to determine the orbital angular momentum l taken in by the neutron in forming this virtual state. The present experiments were performed at thermal energies and could only detect such a resonance for $l = 0$ (see later in chapter). Thus the virtual state is also an S-state.

For the simple potential $V(r)$ there should be spin-degeneracy giving two S-state wavefunctions, namely, $J = 1$, $^3S_1$, and $J = 0$, $^1S_0$. That they are not degenerate is proof of the spin-dependence of nuclear forces and therefore of exchange - but the analysis does not lead to a recipe for the mixture. The ground state is obviously spatially symmetric and spin symmetric; in isospin it must be the antisymmetric $T = 0$ state. The virtual state is spatially symmetric and spin antisymmetric; in isospin it must be the symmetric $T = 1$, $T_z = 0$ state. From the operator forms of the different exchanges (see Appendix C) it is seen that the Majorana term should be the same for both states, but that both Heisenberg and Bartlett terms should change. In the absence of any other states of the deuteron system, no evidence is available on the Majorana term; the effect of the other two terms would appear to be small, but the overall picture is far more complex than presented here so no quantitative

conclusions are drawn.

So far, a simple theory has assigned the characteristics $^3S_1$, $J = 1$, parity even - denoted by 1^+ - to the ground state of the deuteron. An S wavefunction is necessarily symmetric in space, so the deuteron should have no quadrupole moment. But, as stated in point (12) (see p. 38), the deuteron does have a (small) quadrupole moment. There can be no doubt that 1^+ is correct since it is soundly based experimentally. The only other possible wavefunction having 1^+ is the $^3D_1$-state, i.e. the coupling of spin 1 to $L = 2$ ($L = 1$ has odd parity) to give $J = 1$. This wavefunction cannot by itself form the ground state since it has already been shown that no state possessing orbital angular momentum can be bound by any reasonable potential, but in any case it would give too large a quadrupole moment. The remedy is to mix a small amount ($\sim$ 4 per cent probability) of this wavefunction with the basic $^3S_1$ wavefunction. But this mixture no longer has a well-defined value for L; L is certainly a constant of motion for any central force, and so a non-central term must be included in the potential. Such an interaction is already present. The two nucleons have magnetic dipole moments giving a dipole-dipole interaction of the form
$V_{dd} \propto r^{-3}\{3r^{-2}(\underline{\mu}_1\cdot\underline{r})(\underline{\mu}_2\cdot\underline{r}) - (\underline{\mu}_1\cdot\underline{\mu}_2)\}$, i.e.
$\propto r^{-3}\{3r^{-2}(\underline{s}_1\cdot\underline{r})(\underline{s}_2\cdot\underline{r}) - (\underline{s}_1\cdot\underline{s}_2)\} = r^{-3}\,S_{12}$. This magnetic interaction is, however, quite negligible - something of the order of a few hundred eV. It is therefore necessary to introduce a strong interaction of the form $V(r)\,S_{12}$, where $V(r)$, some radial function with a range, replaces the electromagnetic function r^{-3}. This term is obviously different for the parallel

spin orientations of Fig.2.3. Since the deuteron is prolate, the left-hand configuration must lead to an attractive force, giving a negative sign to the potential term above. Note that the large quadrupole moments of heavy nuclei (point (11)) result from correlated motions of a number of nucleons and can arise with central forces between nucleons.

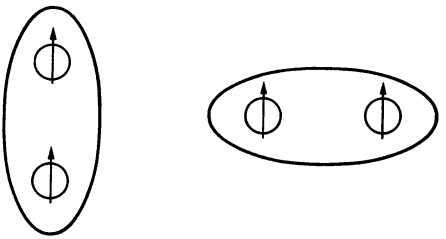

Fig.2.3. Schematic representation of the spin-spin interaction in the deuteron. The spins are shown directed along the symmetry axis.

This admixture of $^3D_1$ in the wavefunction is also required in accounting for the magnetic dipole moment of the deuteron. As stated in point (14), the magnetic moment is almost equal to the sum $\mu_p + \mu_n$; for a pure $^3S_1$-state it should be exactly equal. Classically this statement is obvious; quantum mechanically it is much less obvious. A calculation along the lines of the Landé approach to hyperfine structure in atoms (see e.g. Kuhn 1961) is needed to produce this result and also to determine the magnetic moment of the $^3D_1$-state. Adding the few per cent admixture of $^3D_1$ moves the magnetic moment in the right direction. At this stage a note of caution should be injected.

If the magnetic moments of nucleons derive in large part from the meson clouds, what happens when nucleons are in proximity? For heavy nuclei this could lead to difficulty, but for the deuteron the nucleons are never really close enough for it to matter.

From the fact that the admixture of $^3D_1$ is small, it would be hasty to conclude that the non-central part of the potential is also small compared to the central terms. The 'centrifugal potential' for the D-state for such a small system serves to reduce its effectiveness to such an extent that it becomes important to keep the range of the tensor force at least as large as that of the rest of the potential, in order to keep its magnitude down to an acceptable value. Both the exchange central force and the tensor force can be derived from pion-exchange. The fact that the pion has spin 0 but negative parity means that it must exchange with $l = 1$ in order to preserve nucleon parities. This transfer of orbital angular momentum can cause the spin-flip necessary in some exchange terms and can also produce the tensor term. The exchange of a pion pair (0^+) results in predominantly S-wave transfer and an ordinary force (the combination of two pions gives $T = 2$ or $T = 0$ symmetric and $T = 1$ antisymmetric; these must combine with a symmetric and an antisymmetric space function respectively. Thus $T = 0$, $S = 0$, $L = 0$ is the only possibility for exchange). Since it is also repulsive it is believed to be (partly) responsible for the repulsive core.

NUCLEON-NUCLEON SCATTERING

The (n,p) scattering system at thermal neutron

energies has been cited (p.49) as a source of information on the $^1S_0$ virtual state of the deuteron. Scattering at higher energies with this system and the (p,p) system gives much more information provided that the energy is high enough. Wavemechanically, the projectile is represented by a plane-wave passing the target, placed at the centre of the coordinate system. Such a plane-wave can be analysed into an infinite series of spherical waves converging on and diverging from the coordinate centre and characterized by the value of the orbital angular momentum l. This series is the quantum-mechanical equivalent of the classical sub-division of the wavefront into zones characterized by the impact parameter of the collision; if this latter parameter is p then $pmv = \hbar\{l(l+1)\}^{\frac{1}{2}}$ is the correspondence, where v is the velocity of the projectile of reduced mass m. It is interesting to put numbers into this equation, using the nucleon-nucleon system to define m: for l = 1, 20 MeV incident neutrons in the laboratory system will correspond to a classical impact parameter of 3 fm. The l = 1 partial wave of the plane wave corresponding to 20 MeV neutrons will therefore be unaffected by the collision if the interaction is weak at a distance of 3 fm (the corresponding distance at E_n = 1 MeV is 13 fm). Thus there is a need for higher and higher projectile energies in order to determine the interaction for higher l-values. Although thermal neutrons were sufficient for the S-wave interaction, they could give no information on the interactions at higher l-values for which laboratory energies $\sim$ 30 MeV are necessary. At $E_n \sim$ 300 MeV, the impact parameter is down to 0·7 fm for l = 1, and

so information can be obtained on the potential at close approach.

Note that this same equation with $\{l(l+1)\}^{\frac{1}{2}} = 1$ is also an expression of the uncertainty principle; if the wavelength in the plane wave is λ, then this distance is also the measure of the uncertainty with which a particle can be located and is therefore a limit to which the system can be investigated. Thus for the S-wave interaction an energy of $\sim$ 300 MeV is also required to probe the repulsive core. It is from experiments performed at these energies that the need arose for a repulsive core in the nucleon-nucleon interaction.

Another result can be discerned from a qualitative examination of neutron-proton scattering (see Fig. 2.4). The scattering in the centre-of-mass system

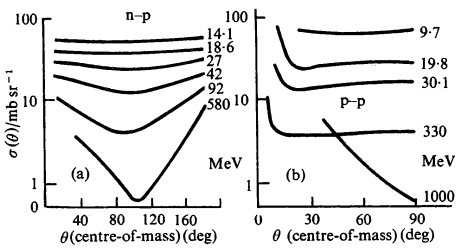

Fig.2.4. (a) n-p scattering at high energies. (b) p-p scattering at high energies. Note, because of similarity of striking and struck particle, the angular distribution must be symmetrical about 90° (centre-of-mass). (Based on HESS (1958). *Rev.mod.Phys.* 30, 368.)

rises sharply toward 0^{o} and 180^{o} at high bombarding energies. Peaking at 0^{o} is to be expected; scattering through large angles requires a head-on collision (or nearly so), whilst small-angle scattering can occur when the neutron passes the proton at a distance and so experiences the extreme tail of the potential. Many more neutrons will 'miss' the proton than 'hit' it. This näive picture conforms with analysis provided the potential is simply $V(r)$, i.e. an ordinary, central potential. The rise at 180^{o} is proof of charge-exchange, corresponding to a small-angle deflection coupled with exchange of charge; the detected neutron, which was originally the proton, moves in the centre of mass in the opposite direction. The type of exchange has been described as charge (or Heisenberg); it could equally have been charge-spin (or Majorana), since the measurements shown refer to unpolarized target and projectile and therefore are averaged over spins. As has been stated previously a large Majorana term is already required.

Another point arising from nucleon-nucleon scattering is that the charge-independence hypothesis appears to be wellfounded. This is not a result which can be arrived at qualitatively; indeed the qualitative conclusion (see Fig.2.4) might well be the opposite. The (p,p) system is, of course, affected by Coulomb forces, and there is an added complication that the two particles are identical, giving rise to interference terms on correctly antisymmetrizing the wavefunction. Even when these differences are taken care of, the comparison of (p,p) and (n,p) is still not like with like: the (p,p) system must have $T = 1$ whilst the (n,p) has equal admixtures of $T = 1$

and $T = 0$. The analysis is lengthy and complicated, but the final result appears to be that the $T = 1$ component of (n,p) behaves similarly to the (p,p) when allowance is made for Coulomb and symmetry effects.

One final piece of information has arisen from nucleon-nucleon scattering with polarized beam and target. A spin-orbit coupling term ($\underline{L}.\underline{S}$) appears to be necessary. Again it has a magnetic analogue, but again it is far too weak. In introducing a strong interaction of this type, the radial function has been given a short range since the term seems to become more effective at higher energies.

SUMMARY

Nuclear forces have been shown to have:

(1) a short range, of order 1·5 fm;

(2) a considerable exchange component, largely of the Majorana type;

(3) a repulsive core, of range $\sim$ 0·5 fm;

(4) a tensor component;

(5) a velocity-dependent spin-orbit interaction.

The greater part of these requirements can be met by a potential derived from exchange of a pion, giving a range of 1·4 fm and accounting for the exchange and tensor components. The repulsive core can be accounted for by simultaneous exchange of two (or more) pions or by exchange of heavier mesons. Among these heavier mesons must be included a vector meson ($J^{\pi} = 1^{-}$) in order to account for the spin-orbit term.

A rough qualitative picture of the nuclear potential is given in Fig.2.5. Notice that in detail

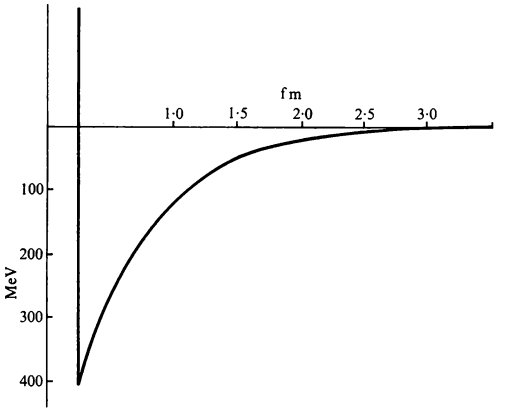

Fig.2.5. An indication of the nucleon-nucleon potential, with hard-core and pion-exchange forces. Note that, in detail, the potential depends upon the combination of T,L,S,J.

it should be different for different combinations of the quantum numbers L, S, and T.

PROBLEMS

2.1. If the n-p potential is of the form $V_1(r) + V_2(r)\underline{s}_1\cdot\underline{s}_2$, show that J^2, parity, L^2, and S^2 are constants of the motion.

2.2. Determine the magnetic moment of the deuteron in the $^3D_1$-state and show that a few per cent admixture of this state into the $^3S_1$-state gives the measured magnetic moment.

2.3. Show that the formal introduction of isospin together with a generalization of the Pauli prin-

ciple does not affect the number of states available to two nucleons.

2.4. The Serber potential is of the form $V(r)(1+P^{S})$ (see Appendix C). Show that it gives no force in odd spatial states.

2.5*. The ground states of $^{10}_{4}$Be and $^{10}_{6}$C lie respectively 0·56 MeV and 3·58 MeV above the 3^{+} ground state of $^{10}_{5}$B. Estimate the energy and spin of one excited state of $^{10}_{5}$B. Comment upon the possibility of exciting this state by deuteron inelastic scattering.

2.6*. Discuss what facts you can deduce about the interaction between two nucleons from the following observations:

(a) in the elastic scattering of neutrons by protons the angular distribution of the scattered neutrons is nearly isotropic in the centre-of-mass system for incident energies below 10 MeV;

(b) when the incident energy is raised above 100 MeV, sharp peaks develop at 0° and 180°;

(c) the binding energy per nucleon in medium-weight nuclei is approximately constant and independent of mass number;

(d) the cross-sections for scattering of slow neutrons by ortho- and para-hydrogen differ by a large factor.

3. Nuclear models

In the previous chapters, simple nuclear models have been introduced, namely, the liquid-drop and Fermi-gas models, which are in effect classical and quantum aspects of the same system. This chapter will discuss three models: the shell model, the collective model, and the optical model. The first is an extension of the Fermi-gas model, whilst the last is a dynamical model concerned with the nucleus at highish excitation, where the restriction of the Pauli principle is decreasing in importance and the system is undergoing transition to something like a liquid drop. In the collective model the restriction to spherical symmetry is removed and the nucleons correlate their motion in the well in such a way as to create the non-spherical well in which they are moving.

THE SHELL MODEL

The atomic shell model has had a great deal of success in accounting for the construction of atoms and the chemical properties of the elements. A brief description is therefore appropriate.

Solution of the Schrödinger equation for the simple system of one electron bound by a central nucleus of charge Z units results in energy levels characterized chiefly by the principal quantum number n. Neglecting the fine structure, each level is degenerate having orbital angular momenta $l = 0,1,\ldots(n\text{-}1)$, which, coupled with spin degeneracy 2, gives a total degeneracy $2n^2$. Also the scale factor of the radial wavefunction

is largely determined by n. In filling up such an atom with its Z electrons, account must be taken of the Pauli principle. Thus the first two electrons go into the lowest energy orbits (n=1) and couple to zero angular momentum; their combined wavefunction is therefore spherically symmetric. The next eight electrons go into orbits with n=2, which are less tightly bound. They are therefore on average much further from the nucleus than the other two, and in addition six of the eight have one unit each of angular momentum which tends to keep them away from the centre. Thus these electrons rarely penetrate the spherical cloud which represents the other two and so move around a central nucleus of effective charge (Z-2). This electron group corresponds to a distribution of charge in the form of a broad spherical shell (in the classical sense), overlapping very little with the shell below. So we build up a complex atom shell upon shell, rather like an onion. This picture is oversimplified and leads to incorrect numbers of electrons in shells ($2n^2$). The reason is that the interpenetration of shells does take place. Because of the centrifugal effect sub-shells of lower l will penetrate the shell below to a greater extent than those of higher l (and the same value of n), so removing the degeneracy enough to move sub-shells effectively from one major energy shell to another. A complete sub-shell, however, is spherically symmetric, so the overall picture is much the same but the numbers of electrons in major shells is now different. This has a profound effect on the chemical properties of the elements.

In trying to adapt this approach to the nucleus,

a number of difficulties arise. In both cases the potential well for the next particle must be built up from the interactions of particles already present. In the atom there is initially a strong attraction to the centre, which carries virtually all the mass and provides a basis for building up layers, in addition ensuring that the layers are physically well-separated. A central force is therefore dominant at all stages, since a uniformly charged spherical shell behaves, in the region outside it, as though the charge is concentrated at its centre. From Chapter 1, however, it is known that there is nothing special about the centre of the nucleus except that it is in a region where the density is constant and where, from symmetry, the average force on a nucleon is zero. Thus the simplest form of potential which might approximate to the average nucleon-nucleus potential is a square well, whilst the next approximation is to make the potential well follow the shape of the density (see Fig.3.1) but extending further out by 1 fm

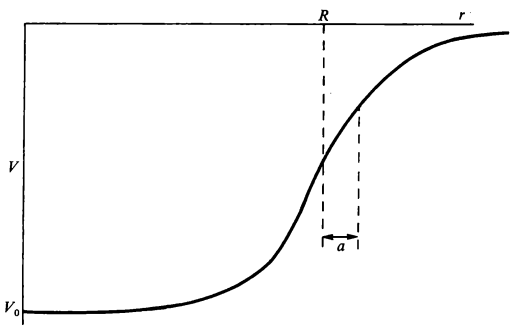

Fig.3.1. Saxon-Woods (or Fermi) approximation to the nucleon-nucleus potential.

or so, to account for the range of the nucleon-nucleon force. With such a well the wavefunctions for the single-particle eigenstates overlap strongly. The centrifugal potential pushes the wavefunction away from the centre for $l \neq 0$, but the effect is not marked for $l < KR$, where K is the wavenumber of the nucleon inside the nucleus of radius R, and $KR \sim 10$ for a medium-weight nucleus.

In addition to greater mixing of wavefunctions enhancing the importance of collisions, there is also the fact that the nucleon-nucleon interaction is short-range so that its contribution to the average potential is much less than its contribution to collision. It would seem improbable that the picture of single-particle orbits in an average central potential should approximate to reality. But so far no account has been taken of the Pauli principle. If one particle gains energy in a collision, then the other particle must lose it. In the ground state of a nucleus all the lowest-lying single-particle states will be occupied, so no particle can lose energy. The same two states must be occupied before and after the collision, so effectively there has been no collision. The mean free path of a nucleon in the ground state of a nucleus is effectively infinite, and the concept of particle orbits is meaningful.

Having established the picture of nucleons in well-defined states it is now appropriate to consider whether these states group into shells, characterized by having similar energies but not necessarily similar spatial configurations as in atoms. In the case of atoms, the periodic table preceded its explanation in terms of the shell model, so for nuclei statistical

analyses of the properties of nuclear ground states revealed the 'magic numbers' which are the nuclear equivalent of the location of noble gases in the periodic table. We take a nuclear property such as binding energy B and plot it against A, or, better still, make a three-dimensional plot against N and Z. Thus in Fig.1.2 (p.19) the measured values of B/A differ from the smooth curve predicted by the mass equation in the direction of increased binding in the region of certain values of N or Z. Statistical analysis reveals the following facts:

1. A plot of binding energy of the last nucleon against N (or Z) reveals discontinuities at certain values of N and Z. (The neutron binding energy may conveniently be determined from the γ-spectrum following capture of thermal neutrons.)

2. Similar discontinuities in the binding of α-particles occur at $N = 82$ and 126.

3. Certain nuclei appear to be naturally more abundant than their neighbours in the plot of stable species. Certain elements have more stable isotopes than their neighbours, i.e. a greater variation of N for a given Z, and correspondingly there are values of N for which a greater variation of Z is permitted.

4. The excitation energies of the first excited states (2^+) of even-even nuclei are at maxima for certain values of N or Z, falling to broad minima between these values. In the broad minima the lowest states (0^+, 2^+, 4^+) form a sequence in which the energy spacings (0,2) and (2,4) are in the ratio 6:14; near the maxima these states

tend to be equally spaced.

5. Quadrupole moments of odd-A nuclei are large when neighbouring even-even nuclei have low-lying first excited states and are near zero when those states lie high.

6. Nuclear reaction cross-sections reveal periodic properties, e.g. thermal-neutron capture cross-sections (as shown in Fig.3.2).

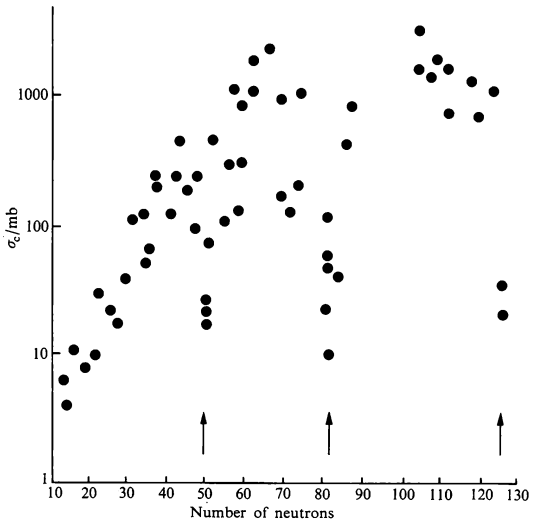

Fig.3.2. Effective capture cross-sections for reactor neutrons plotted against neutron number of target nucleus. Detailed explanation of this curve is complicated, but the influence of shell closure is apparent. (Based on CODD *et al.* (1956). *Prog.nucl.energy* 1, 296.)

7. The parities of nuclear ground states of odd-A

nuclei exhibit a systematic behaviour correlated with the above 'magic' effects.

The nett result of such analyses is that nuclei containing 2, 8, 20, (28), 50, 82, 126 neutrons or protons (apart from 126) are particularly stable (cf. the stability of noble gases). By analogy, shell closure at these neutron or proton numbers is therefore expected. The single-particle energy levels of nuclei should reflect such closure by a larger than usual energy spacing at appropriate places in the level diagram. Simply altering the shape of the central potential through permitted variations is unable to produce energy gaps at these numbers (see Fig.3.3), where the ordering of levels for four types of well is shown to be unsuitable but the addition to the potential of a spin-orbit term ($\equiv V(r)\underline{l}.\underline{s}$ for a given nucleon) achieves success. This suggestion was made independently by Mayer and by Haxel, Jensen, and Suess in 1949. Such a term splits the otherwise degenerate sub-shells $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ obtained by combining the spin and orbital angular momenta of a nucleon.

Since

$$\underline{s}.\underline{l} = \tfrac{1}{2}(j^2 - l^2 - s^2)$$

then

$$\langle \underline{s}.l \rangle = \tfrac{1}{2}\{j(j+1) - l(l+1) - s(s+1)\},$$

giving the value $+\frac{1}{2}l$ for $j = l + \frac{1}{2}$ and $-\frac{1}{2}(l+1)$ for $j = l - \frac{1}{2}$, except for $l = 0$, when both terms are zero. Thus the splitting $\propto (2l+1)$; its increase with l is important for the success of the modification since

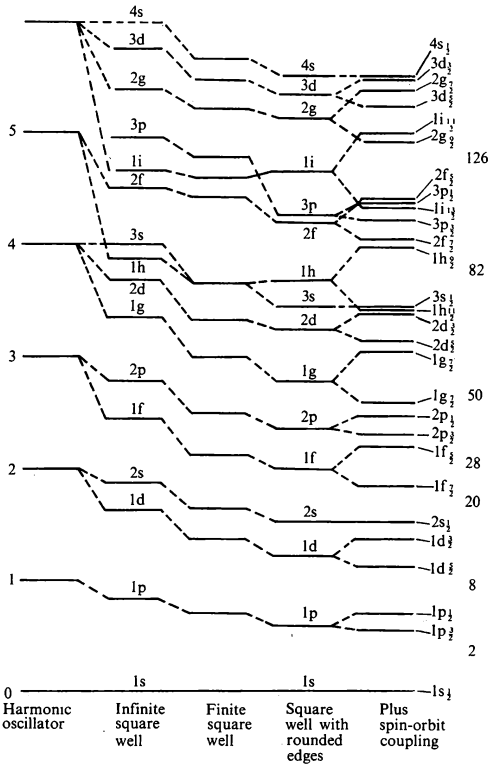

Fig.3.3. Locations of energy levels in four types of well, followed by the effect of a spin-orbit term on the fourth type. Note that the spin-orbit term is necessary to locate the magic numbers, as shown. (Based on FELD (1953). *A.Rev.nucl.Sci.* 2, 239.)

it serves to throw down, from one shell to another, one of the components of the sub-shell with highest l. By making the sign of the spin-orbit coupling opposite

to that arising from a magnetic term (as in atoms) the sub-shell corresponding to $j = (l+\frac{1}{2})$ is correctly thrown down. The order of the filling of neutron shells is then given by the sequence

$$(1s_{\frac{1}{2}})^2 \,|\, (1p_{\frac{3}{2}})^4 (1p_{\frac{1}{2}})^2 \,|\, (1d_{\frac{5}{2}})^6 (2s_{\frac{1}{2}})^2 (1d_{\frac{3}{2}})^4 \,|\, (1f_{\frac{7}{2}})^8 \,|\, (2p_{\frac{3}{2}})^4$$

$$2 \qquad\qquad 8 \qquad\qquad 20 \qquad\qquad 28$$

$$(1f_{\frac{5}{2}})^6 (2p_{\frac{1}{2}})^2 (1g_{\frac{9}{2}})^{10} \,|\, (2d_{\frac{5}{2}})^6 (1g_{\frac{7}{2}})^8 (3s_{\frac{1}{2}})^2 (2d_{\frac{3}{2}})^4 (1h_{\frac{11}{2}})^{12} \,|\, (2f_{\frac{7}{2}})^8$$

$$50 \qquad\qquad 82$$

$$(1h_{\frac{9}{2}})^{10} (3p_{\frac{3}{2}})^4 (2f_{\frac{5}{2}})^6 (3p_{\frac{1}{2}})^2 (1i_{\frac{13}{2}})^{14} \,|\, (2g_{\frac{9}{2}})^{10} (3d_{\frac{5}{2}})^6 (1i_{\frac{11}{2}})^{12} (2g_{\frac{7}{2}})^8$$

$$126$$

. . . . . .

This sequence can be varied slightly by small changes in the spin-orbit potential without affecting the magic numbers.

The notation is slightly different from the atomic convention. The l_j convention is the same but the number preceding does not have the significance of n in the hydrogen-like spectra; it merely numbers the states of given l_j in the order in which they occur (it can be connected with the number of nodes in the radial wavefunction). The superscript is the number of nucleons which the sub-shell can accommodate $(2j+1)$. For protons, because of the influence of the Coulomb potential, the ordering is slightly different; it is the same up to 50 when the next shell looks like

$$(1g_{\frac{7}{2}})^8(2d_{\frac{5}{2}})^6(1h_{\frac{11}{2}})^{12}(2d_{\frac{3}{2}})^4(3s_{\frac{1}{2}})^2$$

and above 82 the sequence is

$$(1h_{\frac{9}{2}})^{10}(2f_{\frac{7}{2}})^8(3p_{\frac{1}{2}})^2 \ \ldots\ldots$$

These sequences, together with the pairing term, which is taken to couple pairs of like nucleons to zero angular momentum whenever possible, account for all the parities of the ground states of stable and nearly stable nuclei and, with a few exceptions, their spins for all even-A nuclei and up to $A \sim 100$ for odd-A nuclei. Two notable exceptions are ^{19}F and ^{23}Na which are $\frac{1}{2}^+$ and $\frac{3}{2}^+$ whereas the sequence of protons predicts $d_{\frac{5}{2}}$, i.e. $\frac{5}{2}^+$ for both. However, the $s_{\frac{1}{2}}$, $d_{\frac{3}{2}}$-states lie in the same shell and it was also discovered that a $\frac{5}{2}^+$ excited state lies low in each nucleus, so it was felt that 'residual forces' (i.e. forces left over after accounting for the well by averaging the nucleon-nucleon interaction) were moving these sub-shells relative to each other. Lately it is considered that nuclear deformation is responsible for these discrepancies.

Above $A \sim 100$ the ground states of odd nuclei do not appear to reflect the order in which the shells should fill; e.g. four isotopes of Sn ($Z = 50$) have ground states $\frac{1}{2}^+$ instead of only one as predicted. The reason lies in the pairing term; adding a second neutron into one of the isotopes $\frac{1}{2}^+$ should produce the pair $(3s_{\frac{1}{2}})^2$, but the pairing of neutrons in a

higher lying state with higher l, probably the $1h_{\frac{11}{2}}$-state, could give an increased binding which more than off-sets the higher energy of the intrinsic state. In which case the pair of neutrons will be found in the $1h_{\frac{11}{2}}$-state in the next even-even isotope, with the $3s_{\frac{1}{2}}$-state unoccupied. So the next odd isotope of Sn will again be $\frac{1+}{2}$, etc.

Also, just below this region, ground states $\frac{7+}{2}$ should be found corresponding to neutron odd numbers 57 - 63. None is found, but in each nucleus where expected, a low-lying $\frac{7+}{2}$ state occurs. A similar explanation can probably be made, though it is possible that the ordering of the $1g_{\frac{7}{2}}$- and $3s_{\frac{1}{2}}$-states should be inverted. For the more massive nuclei, nuclear deformation is expected to occur away from the closed shells, and this would produce a different ordering of the ground-state spins (see next section).

A triumph of the shell model concerns the location of 'islands of isomerism'. It had been noticed that long-lived isomers (excited states of nuclei which decay by γ-emission or its equivalent, internal conversion, with lifetimes long enough to be classified as radioactive) tended to cluster at certain regions of the periodic table. To understand this it is necessary to know something about γ-ray selection rules (see Chapter 5). At this stage we merely need to know that the bigger the spin change the slower the transition. In certain regions of the periodic table, the first excited state is expected to differ in spin from the ground state of an odd-A nucleus by

a large amount. For example, at $Z = 39$ the last proton is in a $2p_{\frac{1}{2}}$-state, whereas the first proton excitation is $1g_{\frac{9}{2}}$; it lies at 0·381 MeV in ^{87}Y, at 0·91 MeV in ^{89}Y, and at 0·551 MeV in ^{91}Y, giving lifetimes of 14 hours, 16 s, and 50 min respectively. Thus it is seen that the spin-orbit term which produces the right sequence of shells also puts sub-shells of high spin in juxtaposition with sub-shells of low spin and so is responsible for isomerism.

Some mention must be made of the magnetic moments of the odd-A nuclei. From the assumption of maximum pairing, the properties of the nucleus should be those of the odd nucleon. The magnetic moment contributed by a single nucleon is

$$\mu = g_l \langle \underline{l}.\underline{j} \rangle + g_s \langle \underline{s}.\underline{j} \rangle \frac{m_{max}}{j(j+1)} \text{ with } m_{max} = j,$$

where $2\langle \underline{l}.\underline{j} \rangle = j(j+1) + l(l+1) - s(s+1)$, and correspondingly for $\langle \underline{s}.\underline{j} \rangle$. This can readily be derived using the vector model (see e.g. Kuhn 1961). By and large the agreement with experimental data is poor. It is best when the odd nucleon immediately precedes or follows shell closure; examples for the odd proton are $^{3}_{1}H$, $^{15}_{7}N$, $^{39,41}_{19}K$, $^{89}_{39}Y$, and for the odd neutron $^{3}_{2}He$, $^{17}_{8}O$, $^{207}_{82}Pb$, but even here $^{41}_{20}Ca$, $^{45}_{21}Sc$, $^{91}_{40}Zr$ show large deviations. $^{19}_{9}F$ might have been quoted above were it not for the fact that theoretical computation has come up with an untidy, three-particle mixture of $s_{\frac{1}{2}}$, $d_{\frac{5}{2}}$, and $d_{\frac{3}{2}}$ wavefunctions outside an ^{16}O core, that happens to give the magnetic moment of an $s_{\frac{1}{2}}$ pro-

ton. The deviations of magnetic moments are as yet unexplained; they could arise from three sources: (1) strict pairing does not occur, (2) nuclei may not be spherical, and (3) the pion field around a nucleon inside a nucleus should differ from that around a nucleon in isolation; this could give rise to a magnetic moment differing from that obtained by vector addition of the individual moments as ascribed to isolated nucleons.

Finally, how does the shell model cope with excited states? The general ordering of levels indicated in Fig.3.3 should represent filled levels up to a certain value and single particle excitations beyond this. If strict pairing does occur, the excited states of an odd-A nucleus should follow this sequence of excitation. Such single-particle excitations do occur, especially when they refer to large l-values in regions of sub-shells of low l. However, the number of excited states up to, say, 3 MeV above ground is usually far greater than the number of single-particle levels available. We are forced to conclude that strict pairing has broken down (it may well take only $\sim$ 1 MeV to break a pair) and that extra states are obtained by the coupling of three or more particles. An extreme example of this is ^{19}F. Its first excited state, at 108 keV, is $\frac{1}{2}^-$ whilst the lowest single-particle states of odd parity $(1f_{\frac{7}{2}})$, $(2p_{\frac{3}{2}})$ should be several MeV higher. Merely decoupling the last two neutrons will not produce odd parity since all three particles are in even-parity orbits - the $s_{\frac{1}{2}}$, $d_{\frac{5}{2}}$, and $d_{\frac{3}{2}}$ which lie close together. It has therefore been assumed that the closed proton p-shell of the ^{16}O core

has been disrupted, and one of the protons has gone into the s-d shell to create something like an α-particle; the $(p_{\frac{1}{2}})$ 'hole' provides the odd parity for this state and others at somewhat higher excitation. Thus, where excited states are concerned, even the closed shells must be considered vulnerable.

The shell model forms a working basis for attempting to determine the level scheme of a nucleus, but there are interactions between the nucleons, termed 'residual forces' since they remain after accounting for the well by averaging out the main body of the forces. These residual forces are often arrived at empirically in building up nuclei progressively from a double-closed shell core by comparison with experimentally determined level schemes.

COLLECTIVE MOTION

In this and the previous chapters, reference has been made to a collective nuclear motion. For this to occur, motion of closed shells is implied, and this in turn means that such shells cannot be spherically symmetric - quantum mechanically there can be no motion about an axis if the system has symmetry about that axis. Thus the occurrence of collective motion is characterized by a departure from spherical symmetry of the nuclear system; it may be the rotational motion of a permanently deformed nucleus or the motion of a nucleus vibrating (possibly about an undeformed mean structure) through different degrees of deformation.

The possibility of permanent deformation was investigated by Nilsson, who calculated the single-par-

ticle states in a spheroidal potential well having a symmetry axis. The distortion of the spherical well is then characterized by the single parameter $\delta = \Delta R/R$, as defined in Chapter 1. For such a well *fixed in space* it must be appreciated that, because of the absence of spherical symmetry, the total angular momentum J is no longer a constant of the motion, but because the system does have symmetry about one of its principal axes the component of angular momentum along this axis, designated by K, is indeed a constant of the motion. The behaviour with δ of the low single-particle levels is shown in Fig.3.4, where the value of $|K|$ is shown on each curve. Since the deformation is quadrupole no distinction is made between up and down, so that each curve has degeneracy 2 corresponding to $\pm K$. In the limit $\delta = 0$, the spherical states appear; in particular the $d_{\frac{5}{2}}$-state appears as the confluence of three states of different $|K|$ and therefore has degeneracy 6, which is in accord with its spatial degeneracy, $2j + 1$.

If the problem is made physically more realistic by isolating the well in space rather than fixing it, then the well itself will be capable of rotation, but the rotation must have no component along the symmetry axis. If the rotational motion is slow compared to the single-particle motion (the adiabatic limit), then the latter will have a well-defined angular-momentum component along the nuclear axis K, but j itself will not be well-defined. The total angular momentum J will also be well-defined (an isolated body) and will have a component K along the nuclear axis and a component M along the external axis of quantization. $\underline{J}$ will be

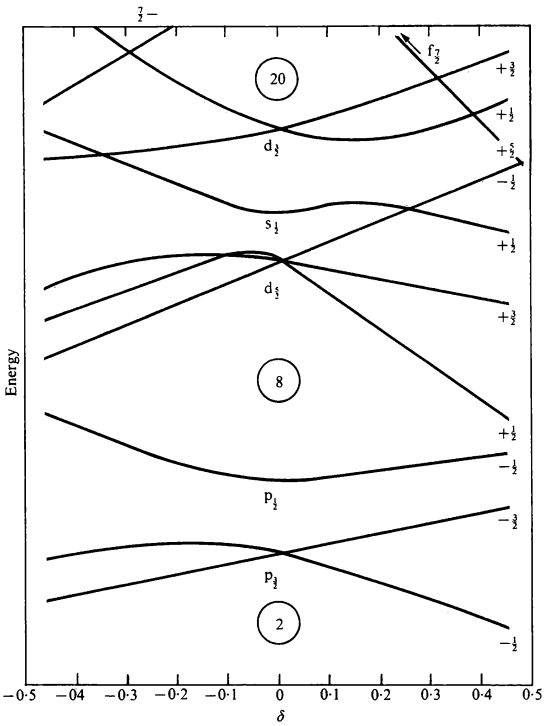

Fig.3.4. Single-particle states in a fixed ellipsoidal potential well, characterized by $\delta = \Delta R/R$. The states are labelled with $|K|$ and parity (they are doubly degenerate) when $\delta \neq 0$. At $\delta = 0$ they correspond to the sub-states of the usual shell-model states. The $^1S_{\frac{1}{2}}$-state has been omitted. (Based on NILSSON (1955). *Dan.mat. -fys.medd.* 29, No.16.)

a vector sum of $\underline{j}$ and $\underline{R}$, the nuclear rotation (see Fig.3.5). The rotational energy will be given by

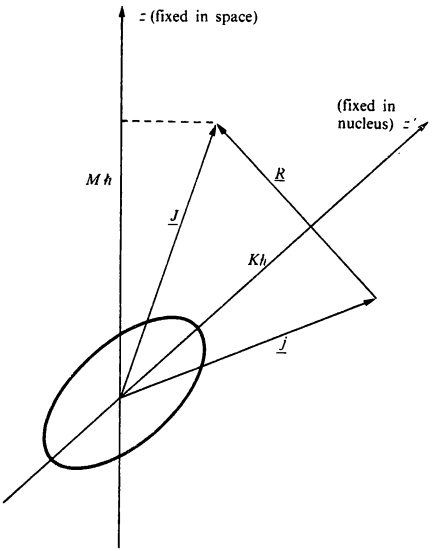

Fig.3.5. Coupling scheme for a deformed nucleus. $\underline{J} = \underline{j} + \underline{R}$, $j_{z'} = J_{z'} = K\hbar$, $J_z = M\hbar$. Note that $\underline{j}$ itself is not a constant of motion, nor is j^2 well defined; e.g. $J^\pi = \frac{7}{2}^-$ can be constructed from $f_{\frac{7}{2}}$, $h_{\frac{9}{2}}$, $h_{\frac{11}{2}}$ orbitals coupled to $R = 0$ or 2, and, except for extremes of coupling, all components will be present.

$$E_R = (\hbar^2/2I)\langle R^2\rangle = (\hbar^2/2I)\langle (\underline{J}-\underline{j})^2\rangle$$

$$= (\hbar^2/2I)(J(J+1) + \langle j^2\rangle - 2\langle \underline{J}.\underline{j}\rangle)$$

$$= (\hbar^2/2I)(J(J+1) + \langle j^2\rangle - 2K^2), \qquad (3.1)$$

since it can be shown that

$$\langle J_x j_x + J_y j_y\rangle = 0,$$

unless there are components of $K \pm 1$ as well as K in the wavefunction. Since there are components of $\pm K$ due to up-down degeneracy, these terms are not zero for $|K| = \frac{1}{2}$, but this case will be ignored. The term in $\langle j^2 \rangle$ can be incorparated into the other single-particle contributions to the energy, and forgotten.

This expression defines a set of rotational states based upon a common wavefunction relative to the nuclear axis. The lowest-lying state will have $J = K$ (the lowest value J can have for a component K along the nuclear axis) and a degeneracy $2K+1$. So, for the isolated nucleus, the confluence at $\delta = 0$ for the $j \equiv d_{\frac{5}{2}}$ configuration will no longer have the correct number of sub-states and would appear to have three values of $J(\frac{1}{2}, \frac{3}{2}, \frac{5}{2})$ instead of the $\frac{5}{2}$ state of the spherical nucleus.

The reason for the discrepancy is that the rotational energy has been ignored in Fig.3.4. How to evaluate this depends upon the nature of the rotation. It could be a rigid-body rotation, or an effective rotation of a hydrodynamical fluid, an example of which is the tidal deformation of the earth by the moon On the surface of the earth this appears like a rotation, but the motion is in fact up and down. Experimental investigation indicates that the latter is nearer the truth and $I \sim I_0\delta^2$, where I_0 is the rigid-body moment of inertia of a sphere, namely, $\frac{2}{5} MR^2$. The rotational energy therefore becomes very large at small δ so the states $J = \frac{1}{2}$ and $\frac{3}{2}$ produced from a $d_{\frac{5}{2}}$ particle move up to very high energies as δ becomes small. More correctly, the coupling scheme breaks down; it is no longer valid to couple the single-par-

ticle motions rigidly to the well and then to allow the whole system to rotate. The figure only approximates to reality when δ is sufficiently large for the rotational energy to be small compared to the spacing of the different particle configurations.

At sufficiently large δ, the diagram correctly predicts the ordering of states. The particles couple strongly in pairs with equal and opposite K, so an even-even nucleus will always have $K = 0$ and a ground state $J = 0$. Putting $J = 0$, $K = 0$ in the rotational energy gives a rotational band of levels of energies, relative to the ground state, $J(J+1)\hbar^2/2I$, where because of symmetry requirements (up $\equiv$ down) only even values of J can occur. This symmetry requirement is met by expressing the wavefunction of the system as a combination of the wavefunctions $(J,M,+K)$ and $(J,M,-K)$ thus: $(\psi^J_{M,+K} + (-1)^{J-K}\,\psi^J_{M,-K})$. For $K = 0$ it is seen that the two components are the same except for the term $(-1)^J$ which eliminates all odd J. For other values of K no such restriction arises, so J increases in unit steps up a band. For odd-A nuclei, the sequence of levels in Fig.3.4 determines a value of $|K|$ for the ground state. A rotational band is now built up having energies relative to ground $\hbar^2\{J(J+1)-K(K+1)\}/2I$. It should be noted that $K=\frac{1}{2}$ produces a more complicated ordering of states which will not be discussed, except to mention that it arises from the $\langle \underline{J}.\underline{j} \rangle$ term in the rotational energy.

A typical level scheme is shown for an even and an odd nucleus in Fig.3.6. It will be seen that rotational states up to large values of J are obtained and that they tend to be lower lying than the next particle configuration. Their locations usually con-

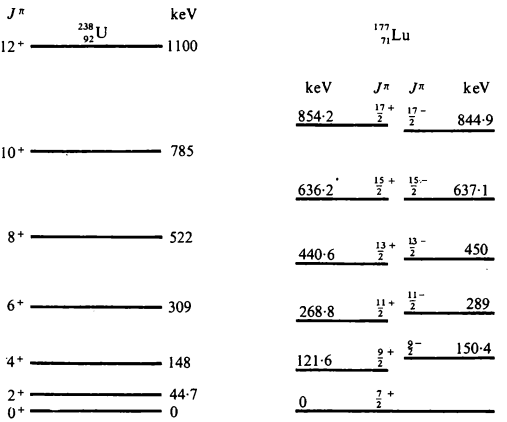

Fig.3.6. Examples of rotational bands in even-A and odd-A nuclei. In the latter nucleus the level schemes representing the two non-interacting bands have been separated laterally. For clarity some levels unconnected with the bands have been omitted from both schemes.

form well with the simple expressions above, but there is usually a small discrepancy which increases with J and which is ascribed to an increase of I due to a centrifugal stretching of the nucleus. Experimentally, levels in a rotational band are revealed by the very large E2 transitions (see Chapter 5) between members - the intrinsic strength can be as large as 200 times single-particle strength in heavy, deformed nuclei in the rare-earth region. Such large enhancements arise from the rotation of the nucleus as a whole, corresponding to a correlated motion of up to Z protons and therefore capable in principle of producing

enhancements up to Z^2.

Finally, nuclei undeformed in the ground state can have excited states corresponding to vibrations through varying degrees of deformation. As in the case of the harmonic oscillator, the different vibrational states are equally spaced, but because the deformation is quadrupole, the vibrational quanta are characterized by 2^+ giving for an even-even nucleus the sequence of states (0^+), (2^+), $(0^+,2^+,4^+)$, $(0^+,2^+,3^+,4^+,6^+)$, etc. where the brackets correspond to 0, 1, 2, 3, etc. vibrational quanta coupled using

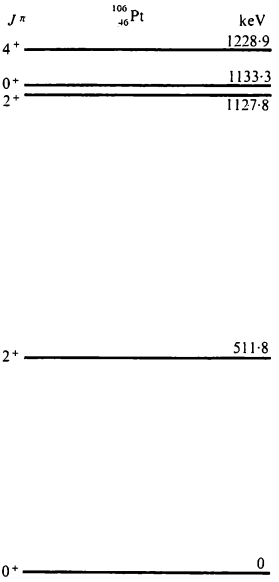

Fig.3.7. A vibrational system. The next levels are located around 1·6 MeV and could include members of the three-quantum vibrational group of states.

Bose statistics. The bracketed states are degenerate in the model, but departure from the model removes the degeneracy as Fig.3.7 shows.

THE OPTICAL MODEL

The previous models refer to a static or quasi-static system since the excited states so far introduced are bound states decaying only by the relatively weak radiation process and therefore having lifetimes much longer than a characteristic nuclear time, which is the time taken for a nucleon to traverse a nuclear diameter. In this situation the quantum-mechanical problem consists of setting up stationary states (just as for the ground state) and then treating the radiative process as a perturbation. When the energy of the system is large enough to permit nucleon (or composite particle) emission then the decay is via the strong interaction so the state can have a lifetime of the order of the characteristic time. The decay process is no longer a small perturbation and a different approach is needed. The optical model meets this need.

This model attempts to cope with the quantum-mechanical problem of the scattering and absorption of particles impinging on a nucleus. Since a simple potential has been used to represent the nucleus in the shell model, it is appealing to do the same for the scattering problem. An argument for the use of such a potential was based upon the long mean free path of a nucleon inside the nucleus as a consequence of the Pauli principle. However, a nucleon coming in from outside has (E_{kin} + 7) MeV (the incoming kinetic energy + binding energy) more energy than any nucleon

within the nucleus. It can collide with such a nucleon thereby losing energy and exciting that nucleon. Because of the Pauli principle it cannot interact with nucleons lying more than (E_{kin} + 7) MeV below the Fermi surface of the nucleus. Thus as the kinetic energy of the incident nucleon increases, the limitations of the Pauli principle are reduced and the mean free path of this nucleon gets shorter.

How is this concept fed into the problem? A beam of particles traversing a region in which the mean free path of a particle is λ will decrease in beam intensity with distance as $I = I_0 \exp(-x/\lambda)$. If the beam of particles, without absorption, is represented by the plane wave amplitude $\sqrt{I_0} \exp(ikx)$, then with absorption it should be

$$\sqrt{I_0} \exp(-x/2\lambda)\exp(ikx) = \sqrt{I_0} \exp\{i(k+i/2\lambda)x\}.$$

The wavenumber has therefore been modified from k to $(k + i/2\lambda)$. But $\hbar k = \{2m(E-V)\}^{\frac{1}{2}}$, where E is the total energy of the particle and V the potential energy. Putting in the modified expression gives

$$k^2 - (1/2\lambda)^2 + ik/\lambda = 2m(E-V').$$

Since the total energy of the particle must be real (it can move outside the region of absorption to be measured), an imaginary component must be added to the potential. Expressing this as $V+iW$ gives $2m(E-V)/\hbar^2 = k^2-(1/2\lambda)^2 \approx k^2$ for reasonably large λ; and $2mW/\hbar^2 = -k/\lambda$ connects the imaginary component with the mean free path.

In its early stages therefore the optical model

used a complex square-well potential, $-(V+iW)$ for $r < R$, zero for $r > R$. This was later changed to the Saxon-Woods potential

$$\frac{-(V+iW)}{1+\exp\{(r-R)/c\}} ,$$

where V, W, R, and c were looked upon as arbitrary parameters in fitting the experimental scattering data at a given energy. The latter two parameters should obviously correspond in some way to the density parameters; the simplest way being to assume that the potential follows the density when allowance is made for the range of nuclear forces. The real term V might be expected to approach the shell-model potential at low enough energy. There is no good reason why V and W should behave in the same way; indeed, an energy-dependence of W is expected from the previous argument which is unlikely to be similar to that of V. Also it can be argued that the bound nucleons near the nuclear surface should be those with greatest energy, for which collisions with the incoming nucleon are least restricted. Thus a peaking of W near the surface may be more realistic, and this can be achieved by associating W with the derivative of the Fermi (Saxon-Woods) function, or by simply using a Gaussian function peaked at the surface. In addition, for treatment of polarization data, it has been found necessary to introduce a spin-orbit coupling term, $\propto \underline{l}.\underline{s}$, where $\underline{l}$ and $\underline{s}$ refer to the incident nucleon.

It is instructive to consider s-wave scattering from a square-well potential of radius R in the limit of very small W. In order to avoid the complexities of determining elastic scattering cross-sections, the

variation with energy of the reaction cross-section is estimated; for $W=0$, there is no absorption and therefore only elastic scattering, but for W small enough the simplifying features of $W=0$ can be used. The wavefunction inside the well (for $W=0$) is of the form $(A/r)\sin Kr$, satisfying the requirement of a node at the origin in r times the radial wavefunction, whilst outside the well the solution can be written $(B/r)\sin(kr+\delta)$, where K and k are the interior and exterior wavenumbers: $K = 2m(E+V)^{\frac{1}{2}}/\hbar$ and $k = (2mE)^{\frac{1}{2}}/\hbar$. The standing-wave solutions indicate that no absorption has taken place, i.e. as many neutrons are moving away from the nucleus as toward it (and the flux of neutrons toward the nucleus $\propto k|B|^2$, though the direct connection is complicated since the neutrons proceeding toward the nucleus constitute a plane wave).

The continuity of the wavefunction and its derivative at the nuclear surface lead to the equations

$$k\tan KR = K\tan(kR+\delta)$$

and

$$A = Bk/(K^2\cos^2 KR + k^2\sin^2 KR)^{1/2}.$$

If for simplicity it is assumed that $K \gg k$ then A/B is seen to vary between maxima of ~ 1 when $KR \sim (n+\frac{1}{2})\pi$ and minima of $\sim k/K$ when $KR \sim n\pi$. Also $|A|^2$ will fall to half its maximum value when $KR \sim (n+\frac{1}{2})\pi \pm k/K$.

If now a very small imaginary term is introduced into the potential, then absorption will take place within the nucleus, but for small enough W this will hardly change the wavefunctions. From the connection

between W and mean free path it is obvious that the reaction yield will be $\propto W|A|^2$. The reaction cross-section will therefore be $\propto W|A|^2/k|B|^2 = Wk/(K^2\cos^2 KR + k^2\sin^2 KR)$, showing maxima and minima with variation of neutron energy. If, as in Chapter 4, the problem were approached in a different way, namely, by inserting the neutron inside the nucleus and observing its transition out, then it will be seen that the maxima correspond to the formation of 'virtual states' of the system. The reaction is said to exhibit 'resonance' when the neutrons are at an energy appropriate to one of the virtual states within the nucleus, cf. forced oscillations of a pendulum near its natural frequency or of an electric circuit. The ratio of resonance half-width to resonance spacing can be estimated from the above equations as $(2k/K) \times 1/\pi$ or $(4k/K) \times 1/2\pi$. The ratio has been expressed in this latter way since the factor $4k/K$ is the mismatch factor at the potential step (see Chapter 4 and Problem 4.1), whilst the factor $1/2\pi$ represents an important theorem concerning widths of virtual states, namely, that on average the decay width of a state into any one channel, after allowing for matching the internal and external wavefunctions, should be $1/2\pi$ times the local spacing of levels with the same spin and parity. For a medium-sized nucleus and a well-depth of 50 MeV, the level spacing of s-wave single-particle neutron states ($J^\pi = \frac{1}{2}^+$ for $J^\pi = 0^+$ targets) is about 15 MeV and the single-particle width about 2 MeV. Thus the single-particle resonances are broad but well separated.

In going to a more realistic model of the nucleus, as well as the single-particle states there are also found vast numbers of levels of the same spins and pari-

ties but corresponding to excitations of more than one particle. In fact the single-particle states alone cannot produce reaction but only re-emission of the same particle. That reaction does take place indicates that the single-particle state is mixed in (by some unspecified perturbation) with all these more complicated states and W must in some way represent the degree of mixing. In the mixed-up system the theorem concerning level width and spacing will on average still hold (see below), though the mixing may reduce the spacing from a few MeV to a few eV.

Inclusion of a large imaginary term in the potential will broaden out the resonance even more (the absorption term is representing other channels into which the state can decay, so the width should roughly increase by the amount W (see Chapter 6)). Since absorption has taken place, followed by decay into another (reaction) channel, a broad resonance is to be expected in this reaction channel, e.g. the emission of inelastic neutrons. Data on total neutron cross-sections are presented in Fig.3.8 and also the results of analysis in terms of the optical model. A feature of the model is that the parameters which describe the potential should vary only slowly from nucleus to nucleus, and this appears to be the case. Typical values for the potential for neutron scattering from a wide range of nuclei at neutron energies up to $\sim$ 50 MeV are

$$V = 52{\cdot}5 - 0{\cdot}6\, E_n \text{ (MeV)},$$

$$W = 2{\cdot}5 + 0{\cdot}3\, E_n \text{ (MeV)},$$

$$R = r_0 A^{1/3}, \text{ where } r_0 = 1{\cdot}20 - 1{\cdot}25 \text{ fm},$$

$$a = 0{\cdot}5 - 0{\cdot}6 \text{ fm}$$

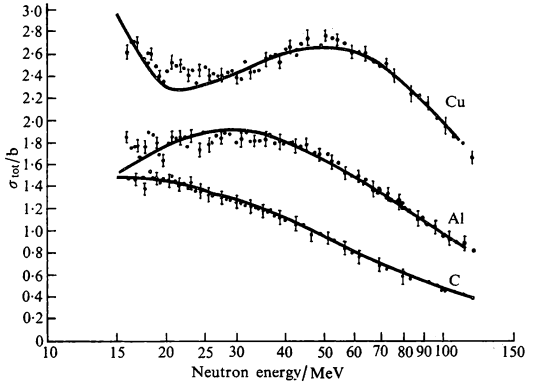

Fig.3.8. Plot of total neutron cross-section against energy for C, Al and Cu. The full lines show optical-model fits. (Based on BOWEN *et al.*(1961). *Nucl.Phys.* 22, 640.)

The success of the model is illustrated better by measurements of angular distributions as shown in Fig.3.9. The situation is paralleled in optics by diffraction through a circular aperture or, better still, the scattering (diffractive) of a small transparent drop of liquid of refractive index different from unity. Also, perhaps paradoxically, the model has had a great deal of success in the analysis of slow neutron data, where compound-nucleus (Chapter 6) resonances of width typically measured in fractions of 1 eV are observed. Representing the nucleon-nucleus interaction by a simple central potential cannot possibly account for these manifestations of the detailed structure of the nucleus, but it has

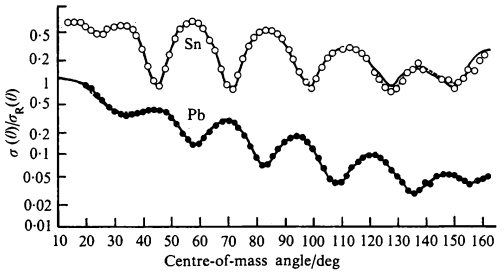

Fig.3.9. Differential cross-section (expressed as a ratio to Rutherford) for elastic scattering of 30·3 MeV protons, showing optical-model fits. (Based on SATCHLER (1967). *Nucl.Phys.* A92, 273.)

been shown that the optical model predicts average cross-sections where the range of averaging covers many compound nucleus resonances. For W very large ($\sim$ single-particle spacing) then the ratio of level width (without barrier) to spacing, which determines the average cross-section, will be of the order of $1/2\pi$; but for W small this ratio, known as the 'strength function', will follow the shape of the optical-model resonance.

PROBLEMS

3.1. The lowest states in $^{207}_{82}Pb$ are (energy in MeV in brackets) $\frac{1}{2}^-$ (0), $\frac{5}{2}^-$ (0·57), $\frac{3}{2}^-$ (0·90), $\frac{13}{2}^+$ (1·63), and $\frac{7}{2}^-$ (2·34). Interpret these in terms of the shell model.

3.2*. $^{9}_{4}Be$, $^{13}_{6}C$, $^{17}_{8}O$ have ground state magnetic moments - 1·17 nm, +0·07 nm, and -1·89 nm respectively. Deduce values for (a) the spins and parities,

(b) the magnetic moments of these nuclei, as predicted by the shell model. Comment on any discrepancies.

3.3*. Under what circumstances are (a) the single-particle shell model, (b) the rotational model expected to give a good description of the properties of nuclei. Describe two features of nuclear behaviour for which either model makes characteristic predictions and show how these predictions can be tested experimentally.

3.4*. The energy levels of the K=0 ground-state rotational band of ^{238}U are given in Fig.3.6 (p.78). Estimate the moment of inertia I by equating the excitation energies with rotational energies. Why does I increase with increasing angular momentum? Why does the sequence contain only even values of J? Compare I with $I_{rigid} = \frac{2}{5} AMR^2$ for the appropriate rigid-sphere rotation and comment on the difference.

3.5. Use Bose statistics to show that three 2^+ vibrational quanta can couple to form the states $J^\pi = 0^+, 2^+, 3^+, 4^+, 6^+$.

3.6. The optical model potential for neutrons on a medium-weight nucleus ($A \sim 125$) is given by the combinations (0,50,2),(30,40,8),(100,25,8), and (300,10,5), where the first number in a bracket is the neutron energy, the second $-V_0$, and the third $-W_0$ all in MeV). Determine the mean free path at each energy near the centre of the nucleus; show that the nucleus is reasonably trans-

parent at each end of the range, and even at maximum absorption there is $\sim$ 10 per cent chance of a neutron passing diametrically through the nucleus. Explain the general trend of V and the mean free path. If the nucleus is transparent for near-thermal neutrons, how is it that they can be captured for long periods of time (see Chapter 6)?

3.7. Work through the problem outlined on p.83 and verify the two equations given there.

4. Spontaneous decay of Nuclei I: α and ß decay

In Chapter 1 it has been shown that the stable nuclei are surrounded by neighbours which are unstable to ß-decay, whilst further away from the stability line nucleon emission can take place. These latter nuclei can only be observed in nuclear reactions. For mass values greater than 150, even the nuclei on the stability line become unstable, not to nucleon emission but to emission of α-particles because the binding energy per nucleon of the α-particle is similar to that of a nucleus.

α-DECAY

Historically, α-particle decay was observed in the heavier nuclei ($A > 206$), and the lifetime for decay was seen to be very strongly correlated with the energy of the α-particle emitted (because the process is a break-up into two particles, the energy of the α-particle is well defined for unique parent and daughter states). For these nuclei all observed α-emissions lie within the energy limits $4\ \text{MeV} < E_\alpha < 9\ \text{MeV}$, the lower value being associated with a lifetime $\sim 10^{+18}$s and the upper value with 10^{-6}s, according to the Geiger-Nuttall rule which can be expressed as $\ln \lambda = a + b \ln E$, where λ^{-1} is the mean life. This rule explains why, although almost all nuclei in the range $A = 150 - 200$ are unstable to α-emission, only a few have been observed to decay in this way; in general the available energy is very low and the lifetime too long to permit detection of the process. An example of an observed decay in this

region is $^{152}_{64}$Gd, with a lifetime of 10^{14} years and an α-particle energy 2·24 MeV. These figures are not compatible with the Geiger-Nuttall relationship as applied to the natural radioactive series, but it will be seen later that the connection between lifetime and energy depends very strongly on the nuclear charge Z.

Another historic experiment was the bombardment of the daughter nucleus with α-particles of the same energy as (or even greater than) those emitted by the parent. The scattering was observed to be of the pure Rutherford type, which, on the face of it, indicated that the nucleus was still behaving effectively as a point in its interaction with an α-particle approaching it from outside, even though this particle was closer to it than the region from whence (in the classical sense) an α-particle must have originated in the decay of the parent. This paradox was explained by Gamow in his theory of barrier penetration.

In Fig.4.1 an idealized form is presented of the

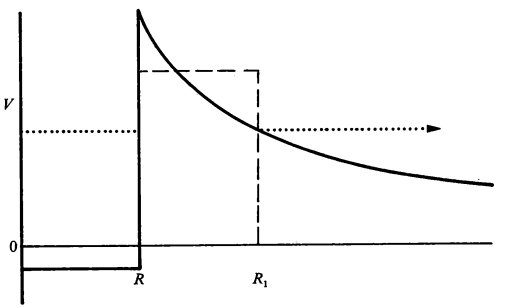

Fig.4.1. Idealized potential for s-wave α-particle plus product nucleus. Also shown is the potential step approximation.

potential to which an incoming or outgoing α-particle is exposed. The shape of the potential is dominated by the Coulomb interaction at long distances and by nuclear forces at short distance. These latter have been simplified to a square-well form, though they should conform to something like the optical potential, which will smooth out the sharp discontinuity and also, by means of the imaginary term, produce absorption within the nucleus. Also shown in Fig.4.1 is a cruder approximation to the potential in terms of potential steps.

It is instructive to solve the wave-mechanical problem, in one dimension, of a plane wave of wavenumber K impinging from the left onto this latter potential, being partially transmitted, and finally emerging as a wave of wavenumber k in the far right region; in the intermediate region it must be represented by real exponentials (multiplied by complex amplitudes) if, in this region, its total energy is less than the height of the potential step. This exercise is left to the reader.

If in this problem the incident wave is of the form $I\exp(iKr)$ and the transmitted wave $A\exp(ikr)$ then the transmission coefficient T is defined by the ratio of transmitted to incident flux

$$T = v_0|A|^2/v_I|I|^2 = k|A|^2/K|I|^2,$$

where v_I, v_0 are ingoing and outgoing velocities. For this particular problem

$$T = \frac{16Kk\gamma^2}{(K^2+\gamma^2)(k^2+\gamma^2)} \exp(-2\gamma\Delta), \qquad (4.1)$$

where $\gamma = \{2m(V-E)\}^{1/2}/\hbar$ and Δ is the width of the barrier, (R_1-R).

There is a direct correspondence between the solution of a one-dimensional problem and the radial dependence of the solution of the corresponding three-dimensional problem with spherical symmetry. The one-dimensional wavefunction $U(r)$ is simply connected to the radial dependence of the three-dimensional wavefunction $F(r)S(\theta,\phi)$ by $U(r) = rF(r)$, but the potential for the one-dimensional problem is not just $V(r)$ but $V(r) + l(l+1)\hbar^2/2mr^2$, i.e. allowance must be made for the centrifugal potential. Thus the solution above is appropriate for transmission of an $l = 0$ α-particle through the spherically symmetric potential step. Also the transmission coefficient is unchanged since the flux at r is $v(r).r^2|F(r)|^2$ in the three-dimensional case and therefore $\propto v|U|^2$ as before. It has been assumed that $S(\theta,\phi)$ is correctly normalized, i.e. $\int|S(\theta,\phi)|^2 d\Omega = 1$.

However, considered as a three-dimensional problem, there must be a node (in $U(r)$) at the origin. In principle, no simple solution as obtained can give such a node since, to conserve flux, the reflected wave on the left (i.e. inside the spherical potential) must have smaller intensity than the incident wave. But the difference ($\sim \exp(-2\gamma\Delta)$) may be very slight, and to good approximation can be ignored. The requirement of a node at the origin in addition to the (amplitude and) phase relationship between the two components derived from solution of the previous problem overdetermines the solution within the potential, leading to a discrete set of solutions corresponding to energy levels of the α-particle. These solutions

are not quite stationary states since they are decaying via transmission through the barrier.† However, if $\exp(-2\gamma\Delta)$ is small then the decay is slow when measured on a nuclear time scale. The decay constant λ_α (in s^{-1}) will be the outgoing flux when the wavefunction inside the 'nucleus' of radius R is normalized to unity: this normalization is realized by putting

$$\int_0^R |I\exp(iKr) + J\exp(-iKr)|^2 dr = 1,$$

where $J\exp(-iKr)$ is the reflected wave. Because of the node at the origin the integral becomes $\int_0^R 4|I|^2\sin^2 Kr\ dr$ giving $|I|^2 \approx 1/2R$, since the point R is very close to being an antinode. From the uncertainty principle,

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†A more accurate solution of the problem introduces difficulties of normalization. The outgoing wave must slowly increase in amplitude as the distance from the nucleus increases, representing the fact that, at an earlier time, the probability of finding an α-particle within the nucleus was greater, and therefore the leakage through the barrier was greater. Thus the normalization integral diverges outside the nucleus. The approximate solution given can be looked upon as a perturbation approach in which an infinite confining potential has been switched off at $t = 0$ and the rate of increase of the final state (corresponding to a free α-particle) is determined when the initial state is known to represent one α-particle confined within the nucleus.

$$\Gamma_\alpha = \hbar\lambda_\alpha = \hbar v_0 |A|^2 \quad \text{when} \quad |I|^2 = 1/2R,$$

$$\text{i.e.}\,\Gamma_\alpha = \frac{\hbar^2 k}{m} \cdot \frac{|A|^2 K |I|^2}{K|I|^2} = \frac{\hbar^2}{m} \cdot T \cdot \frac{K}{2R} = \frac{\hbar^2}{2mR^2}\,(KRT). \qquad (4.2)$$

This expression separates the α-decay width into two parts: the part $\hbar^2/2mR^2$, which has dimensions of energy (width) and is the nuclear scale factor for width, and the dimensionless bracketed term, which is the transmission coefficient suitably modified for an effectively bound α-particle rather than a flux. This transmission factor is the product of a major term, the barrier penetrability, and a minor term which here represents the product of two mismatch factors arising from the two discontinuities in the potential (cf. reflection of optical waves at a glass surface or vibrations at a joint between stretched strings). In a more realistic calculation, introducing the Coulomb potential, it is reasonable to expect (and more detailed calculations confirm this) the penetrability term to be replaced by $\exp\{-2\int_R^{R_1}\gamma(r)\,dr\}$, since γ is now a varying function of r. The limits of integration R and R_1 are the points where $\gamma(r) = 0$. The slowly varying γ effectively changes adiabatically to an equally slowly varying k without discontinuity and so eliminates the mismatch factor at R_1. At R the discontinuity of the nuclear square-well potential will introduce a mismatch factor, but a (rapidly changing) Saxon-Woods-type potential will reduce the mismatch though not necessarily eliminate it. This accounts for the fact that analysis in terms of a square well appears to require a potential of approximately zero inside the nucleus whereas analysis in terms of the optical model usually gives a negative

internal potential (-25 MeV or so), though with considerable latitude since there are other variables in the analysis. Note that the modified barrier penetrability may also be applied to the case $l \neq 0$, since $\gamma(r)$ will be computed with the centrifugal barrier included.

The chief source of uncertainty in the calculation of decay widths and lifetimes is in deciding what fraction of the single particle width $\hbar^2/2mR^2$ must be used for real nuclei. The original calculations employed the full single-particle value, as given here, which for nuclei with $A > 200$ is about 0.1 MeV, though the precise value will depend on the nuclear potential adopted. It was later suggested that 1 eV may be more realistic; this estimate was based upon widths of resonances in slow neutron reactions (see Chapter 7). Such a large reduction in nuclear factor was off-set in the barrier-penetrability term by employing a larger nuclear radius. Nowadays, these radii so deduced are considered to be too large, which is to be expected since the levels from which the width was inferred are at an excitation of order 7 MeV and statistical theories of nuclear levels require that as level densities increase then level widths (for a particular channel, though not necessarily total widths) should diminish. Somewhere near 1 per cent of the single-particle width is now considered a reasonable estimate. Obviously, in face of such uncertainties, variations in mismatch factors due to details of the nuclear potential are of little significance.

An examination of the α-spectrum of a given radioactive nucleus reveals in general more than one

transition energy. The ground state of the parent nucleus can decay to a number of states in the daughter nucleus (of which the ground state may be one). In general too the higher the excitation of the daughter state the smaller the intensity of the transition, since the penetrability is so strongly dependent upon the α-energy. For the naturally occurring α-emitters a rough estimate is a factor of 3 decrease in transition probability for 100 keV decrease in α-energy. In addition, the angular momentum requirements will contribute to intensity variations. Table 4.1 gives a rough guide to this effect.

TABLE 4.1

l	0	1	2	3	4	5	6	7	8
Reduction factor	1	0·7	0·37	0·14	0·04	7×10^{-3}	1×10^{-3}	1×10^{-4}	7×10^{-6}

As an example consider the decay $^{238}_{94}\mathrm{Pu} \rightarrow {}^{234}_{92}\mathrm{U} + \alpha$. Table 4.2 gives the measured transitions to states of known J (since $^{238}_{94}\mathrm{Pu}$ has $J = 0$ in the ground state, the final J value is also the l-value for the emitted α-particle).

The first five transitions are to members of a rotational band. Such states have similar wavefunctions since they represent the same nuclear configuration in different degrees of rotation; it is therefore gratifying to see that the decrease in transition strength is largely accounted for by the variation of penetrability with energy and angular momentum. The

TABLE 4.2

J	0^+	2^+	4^+	6^+	8^+	1^-
E(keV)	0	43	143	297	499	786
Relative strength:						
Estimated	1	0·25	9×10^{-3}	3×10^{-5}	3×10^{-8}	1×10^{-3}
Observed	1	0·39	$1{\cdot}2\times10^{-3}$	7×10^{-5}	1×10^{-8}	2×10^{-7}

sixth transition (others are given in the literature for this decay, but this one is typical) has a much weaker strength than would be accounted for by these means. It therefore reflects the nuclear contribution to the process. Thus the conclusion to be drawn is that for members of a rotational band (or for that matter a vibrational band) the simple estimates of relative strengths should be fairly good, but in general the dominant factor is the nuclear intrinsic width when low-lying states are being compared.

Finally, mention must be made of 'long-range' α-particles (this historical term is synonymous with 'high-energy'). An excited state can, in principle, emit α-particles; indeed its decay probability into the α-channel will probably be greater than that of the ground state. However, it will be competing unfavourably with γ-emission (see Chapter 5) at least for states near ground; at high enough excitation the reverse will occur but such states are not excited in the natural α-decay series. If the energy available to the α-particle is very large

then the decay probability will be correspondingly large. If also the γ-transition probability is reduced because of high spin change, for example, then it may be possible that the excited state can have an observable α-decay. The key word is observable; modern techniques have revealed quite large numbers of 'long-range' transitions giving low-intensity α-groups above an intense ground-state group. From what has been said they are most likely to be found in conjunction with short-lifetime ground-state decays. An example is $^{212}_{84}$Po, with a ground-state half-life of 304 ns and E_α = 8·79 MeV, having three such transitions, of which the most intense, $1{\cdot}8 \times 10^{-4}$ of the ground state strength, is from a state at an excitation of 1·8 MeV.

β-DECAY

In Chapter 1, β-decay has been introduced as a process involving the simultaneous emission of a β-particle and a neutrino. The neutrino was postulated to satisfy the requirements of conservation of energy, momentum and angular momentum in this process. The β-spectrum is continuous up to an upper limit which is the energy expected for the β-particle in the absence of an undetected particle. Similarly experiments involving the direction of recoil of the daughter nucleus indicate that momentum is not conserved - or preferably that an undetected particle has also been emitted. Also, to conserve angular momentum a neutron cannot decay into two spin-$\frac{1}{2}$ particles, whether isolated or inside a nucleus. The neutrino must be neutral since charges balance without it. It

must have intrinsic spin $\frac{1}{2}$ and a very small rest mass, since the β-energy at maximum accounts very closely for all the energy available. Efforts were made in the past to detect the neutrino, but failed. Now that recent efforts have been successful, it is not surprising that former methods failed since it is now known that neutrinos can pass through the Sun, the centre of which is a prolific source - indeed neutrino emission accounts for a considerable fraction of the energy emitted by stars.

A simple theory of β-decay was provided by Fermi, who exploited the similarity with emission of radiation. A proton (or neutron) changes into a neutron (or proton) by simultaneously creating the β-ν pair just as an excited state of an atom changes to a lower state by creating a photon. Note that in isolation a proton cannot decay to a neutron because it has less mass, but inside a nucleus a bound proton may be in a state of higher energy than an equivalent bound neutron, so the process can take place; if the total masses are favourable then the reaction can go.

Considering first the decay of an isolated neutron ($n \rightarrow p + e^- + \bar{\nu}_e$) then the wavefunctions of the two light particles can be represented by plane waves in the directions of emission, $\psi_\beta = N_\beta \exp(i\, \underline{k}_\beta . \underline{r}_\beta)$ and $\psi_\nu = N_\nu \exp(i\underline{k}_\nu . \underline{r}_\nu)$. The origin of the coordinate system is assumed to be the location of the nucleon both before and after the event (i.e. recoil has been neglected for simplicity). The plane-wave approximation should be adequate for the neutrino, but for the β-particle the long-range Coulomb interaction with the proton (or nucleus when at a later stage the nucleons are considered to be part of a nucleus) will

distort the wave - this point will be taken up later. In the absence of guidance, Fermi assumed the simplest possible interaction namely that it is proportional to the simultaneous overlap of all four particles, i.e.

$$H_{fi} = g\int \psi_f^* \psi_i \, d\tau = g\psi_\beta^*(0)\psi_\nu^*(0)$$

in this simple case of point nucleons at the origin. From perturbation theory (Fermi's golden rule) the transition probability per unit time is

$$P = \frac{2\pi}{\hbar}|H_{fi}|^2 \frac{dn}{dE} ,$$

where dn/dE is the density of final states. In applying this formula it is convenient to enumerate the final states when the system is confined to a specified (large) volume Ω. The normalization factors for the plane waves, N_β and N_ν, are then both $1/\Omega^{\frac{1}{2}}$, and the density of plane-wave states of a particle having momentum between p and $(p+dp)$, with the particle anywhere in Ω is $p^2 dp\Omega/2\pi^2\hbar^3$. The number of states for the β-momentum in the range p_β to $(p_\beta+dp_\beta)$ and the ν-momentum in p_ν to $(p_\nu+dp_\nu)$ is then

$$dn = (p_\beta^2 dp_\beta/2\pi^2\hbar^3).(p_\nu^2 dp_\nu/2\pi^2\hbar^3)\Omega^2.$$

In multiplying these two functions together it has been assumed that there is no correlation between the directions of emission of electron and neutrino. This is partly because of the initial simple assumptions, but chiefly because the final nucleon is available to take up the requisite momentum required for conservation; because it is so massive it can do this and still

have little effect on the energy balance. It does not appear directly in the calculations, but its presence has been felt nevertheless.

To determine dn/dE it is necessary to transform from variables p_β, p_ν to p_β, E, using the equation

$$E = c(p_\nu^2 + m_\nu^2 c^2)^{1/2} + E_\beta,$$

where E is the total available energy. The transformation is

$$dp_\beta dp_\nu = \frac{\partial p_\nu}{\partial E}\, dp_\beta dE = \frac{1}{v_\nu}\, dp_\beta dE = \frac{1}{c}\, dp_\beta dE,$$

since the rest mass of the neutrino is either zero or close to it. Very close to cut-off of the β-spectrum this neglect of the rest mass of the neutrino could distort the spectrum, but such effects have not been observed. Thus the density of states is given by

$$\frac{dn}{dE} = \frac{\Omega^2 p_\beta^2 p_\nu^2}{4\pi^4 \hbar^6 c}\, dp_\beta = \frac{\Omega^2 p_\beta^2 (E-E_\beta)^2}{4\pi^4 \hbar^6 c^3}\, dp_\beta,$$

using $E_\nu = cp_\nu = E-E_\beta$.

The transition probability per unit time to produce electrons in the momentum range p_β to $(p_\beta+dp_\beta)$ is

$$P(p_\beta)dp_\beta = \left(\frac{g^2}{2\pi^3 \hbar^7 c^3}\right)(E-E_\beta)^2 p_\beta^2 dp_\beta, \tag{4.3}$$

or in terms of energy

$$P(E_\beta)dE_\beta = P(p_\beta)\,\frac{dp_\beta}{dE_\beta}\,dE_\beta = \left(\frac{g^2}{2\pi^3 \hbar^7 c^3}\right)(E-E_\beta)^2\,\frac{p_\beta^2}{v_\beta}\,dE_\beta. \tag{4.4}$$

Note that the result is independent of Ω, the volume in which the system was confined. In the non-relativistic approximation the energy spectrum is proportional to $(E - E_\beta)^2\sqrt{E_\beta}$, and this accounts for the general shape of a β-spectrum.

To extend this treatment to complex nuclei it is necessary to consider the decay of each nucleon as an independent event and to sum the amplitudes of each event. So far the interaction is at a point, so the β-ν combination cannot take away orbital angular momentum; since no mention has been made of spin the combination must be assumed to be the spin=0 state. In the nucleus the decaying nucleon merely changes its charge but remains in the same nuclear state - the instantaneous effect can be represented by $\sum_k \tau_k^{\pm}\psi_i$, where $\tau_k^{\pm}$ is the isospin raising or lowering operator acting on the kth nucleon in the nuclear wavefunction ψ_i (τ^- for β^- emission, τ^+ for β^+). This instantaneous wavefunction must be projected onto the final wavefunction to determine the nuclear contribution to the process, giving the matrix element

$$M_{fi} = \int \psi^* \sum_k \tau_k^{\pm}\psi_i \, d\tau .$$

The summation may be taken over all nucleons since τ^- acting on a proton gives zero - it changes neutrons to protons - and correspondingly for τ^+. Obviously, from what has been said above, the selection rule $\Delta J=0$, with no parity change, holds for the nuclear states.

Extending the theory to include the possibility of the β-ν combination taking away one unit of spin leads to a parallel decay obeying the selection rule $\Delta J=0, \pm 1$, with no parity change (but $J=0\rightarrow 0$ excluded).

The nuclear matrix element must contain a spin operator capable of changing the spin orientation of the nucleon k - it must be in the form of a vector in order to take away (spin) angular momentum. There is no need to go into its form here; the simple picture of the β-ν combination being emitted in a wave having $L=0$, $S=1$ is sufficient to arrive at the above selection rule. The spinless emission is known as the Fermi process and the spin-1 emission is ascribed to Gamow-Teller. The processes so far discussed are termed 'allowed'.

The treatment of nucleons as point particles is an approximation which allowed the β-ν combination to take away no orbital angular momentum. Whilst there is little to be gained by giving the nucleon structure in the decay of an isolated neutron (to preserve parity a two-unit change would be necessary and this is unlikely for such a small structure), it should be noted that even for point nucleons it may be possible to change the orbital angular momentum of a nucleus. If the emitting nucleon is at the periphery of the nucleus then, classically, its recoil momentum will produce a change of angular momentum. The greatest change will occur when an electron of maximum energy is emitted tangentially from the surface, and will be of order $(E_\beta/c)R_N$. Taking $E_\beta \sim 1$ MeV (for which $E_{\beta/c} = p_\beta$ is only approximate) and $R_N \sim 6$ fm (a medium-size nucleus) gives $0{\cdot}03\hbar$ for the classical angular momentum in this extreme situation. Thus the taking away of one unit of angular momentum must be looked upon as possible but improbable. Mathematically, the two plane waves of β and ν are amalgamated to a single plane wave $\exp(i\underline{k}.\underline{r})$, where $\underline{k} = \underline{k}_\beta + \underline{k}_\nu$, and, since

the interaction is still assumed to be at a point, $\underline{r}_\beta = \underline{r}_\nu = \underline{r}$. This plane wave is expanded within the confines of the nucleus giving $\exp(i\underline{k}.\underline{r}) = 1 + i\underline{k}.\underline{r} - \frac{1}{2}(\underline{k}.\underline{r})^2 \ldots$. The matrix element for the process will now become

$$H_{fi} = g\int\psi_f^* \sum_k \tau_k^{\pm}(1+i\underline{k}.\underline{r}_k\ldots)\psi_i d\tau,$$

where for convenience the possible spin factor is omitted. If ψ_f and ψ_i have opposite parities then the leading term in the bracket will integrate to zero, but the next term, which is itself odd, may not do so. If polar axes are chosen such that $\underline{k}$ lies along z then $\underline{k}.\underline{r}_k$ becomes $kr_k\cos\theta$. The angular wavefunctions of nucleon k in ψ_f and ψ_i are in the form of spherical harmonics $Y_{l,m}(\theta,\phi)$, and their properties are such that $\int Y^*_{l_f,m_f}(\theta,\phi)\cos\theta\ Y_{l_i m_i}(\theta,\phi)d\Omega$ will be zero unless $m_f = m_i$† and $l_f = l_i \pm 1$. Thus the kth nucleon changes angular momentum by one unit (cf. emission of a photon by an electron in an atom) and, by recoupling to the rest of the nucleus, the selection rule follows: $\Delta J=0, \pm1$, with change of parity ($J=0 \not\to 0$ excluded) for the Fermi process. By coupling up to $S=1$, the corresponding Gamow-Teller selection rule is $\Delta J=0, \pm1, \pm2$, with change of parity.

Consideration of the radial integral indicates that these transitions will be down by a factor $(kR_N)^2$ on the allowed transition - the k^2 term arises from the integral and the R_N^2 from the fact that an integral $\int\psi_f^* r\psi_i r^2 dr$ must be $\sim R_N\int|\psi_f||\psi_i|r^2dr$, and we assume that all integrals of the latter type have approximately the same value. These types of transition are known as 'first forbidden' since they are down by the

†A different choice of axes results in $m_f = m_i \pm 1$.

factor $(kR_N)^2$, which, as previously estimated, is $\sim 10^{-3}$ or 10^{-4}. The next term in the series would produce second forbidden transitions and will be down by the same factor again.

Before examining whether this classification conforms with experimental fact, it is necessary to consider the effect of nuclear charge on the process since this will vary from nucleus to nucleus. The wavefunction of a particle in a Coulomb field is complicated, but here we merely need to know the behaviour at $r = 0$, where the process occurs, and $r = \infty$, where the products are observed. If at infinity the wavefunction for the β is the original plane wave then $|\psi_\beta(0)|^2$ will be multiplied by the factor $F(Z,E) = 2\pi\eta\{1-\exp(-2\pi\eta)\}^{-1}$, where $\eta = \pm e^2/4\pi\varepsilon_0\hbar v_\beta$ (+ for electrons, - for positrons), v_β being the speed at infinity and Z the atomic number of the final nucleus. The Coulomb correction enhances the probability of electron-emission and decreases that of positron emission, especially at low energies. In the latter case it is effectively the Gamow barrier-penetrability factor since the positron may well be created with negative kinetic energy and need to penetrate the barrier before emerging with positive energy. The enhancement of the β^--process occurs because the attractive potential increases $|\psi_\beta(0)|^2$ relative to $|\psi_\beta(\infty)|^2$. The normalization of the wave is largely determined by $|\psi_\beta(0)|^2$ since Ω is very much larger than the nuclear volume, but the transition probability depends on $|\psi_\beta(0)|^2$. The effect of the Coulomb potential on $\beta^\pm$-spectra is shown in Fig.4.2.

With the Coulomb correction the momentum spectrum

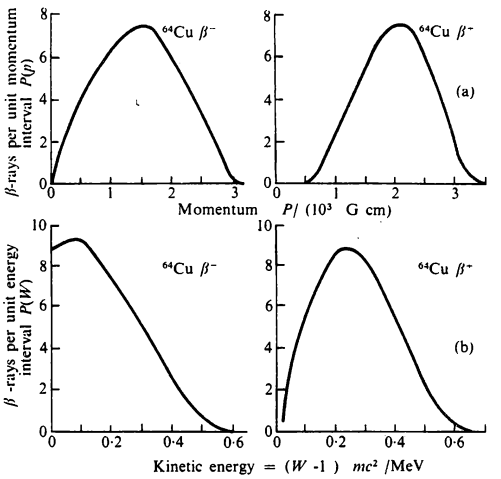

Fig.4.2. Effect of Coulomb potential on β-spectra for ^{64}Cu decay, which shows β^- and β^+ (and electron-capture) decay. Note that the β^- energy spectrum has finite intensity at zero kinetic energy, but the momentum spectrum does not (the effect of the relationship $dE = v\,dp$). (Based on EVANS (1955). *The atomic nucleus*, McGraw-Hill, New York.)

for allowed transitions is of the form

$$P(p_\beta)\,dp_\beta = CF(Z,E_\beta)(E-E_\beta)^2 p_\beta^2 dp_\beta, \qquad (4.5)$$

where the constant C contains terms independent of the β-energy. Thus a plot of $\{P(p_\beta)/Fp_\beta^2\}^{1/2}$ against E_β is a straight line intersecting the abscissa at E. This, or its equivalent for the energy spectrum, is known as a Kurie plot, an example of which is shown

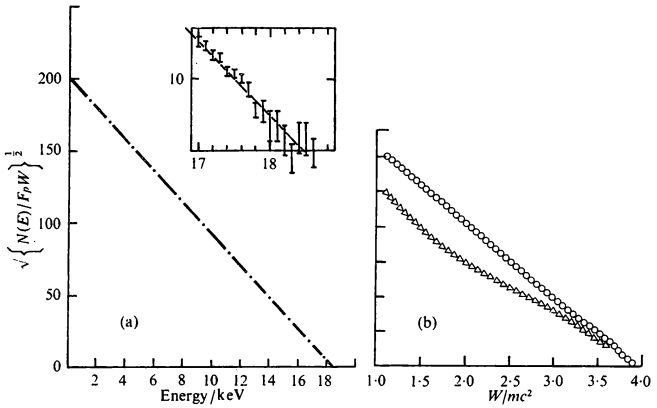

Fig.4.3. (a) Kurie plot for allowed decay of tritium. Inset shows fit near end point. Lack of deviation indicates that the rest mass of the neutrino must be small. (Based on LEWIS (1970). *Nucl.Phys.* A151, 120.)
(b) Kurie plot for first forbidden decay of ^{89}Sr. The curved plot is the usual plot with F_0 in the ordinate function; the linear plot contains a theoretical correction for the first forbidden transition. (Based on WOHN and TALBERT (1970). *Nucl.Phys.* A146, 33.)

in Fig.4.3. The linear plot results for allowed transitions only; in forbidden transitions terms in $k^2 = k_\beta^2 + k_\nu^2 + 2\underline{k}_\beta.\underline{k}_\nu$ appear, which depend upon the relative direction of the β-ν. On integrating over all directions, the term $\underline{k}_\beta.\underline{k}_\nu$ averages to zero, and the two remaining terms inject a factor $\{p_\beta^2+(E-E_\beta)^2/c^2\}$ into the momentum spectrum for a first forbidden transition. The Kurie plot will not be linear, but a modified Kurie plot, including this factor, can

restore the straight line. This technique can be used to determine the type of decay (see Fig.4.3).

A convenient classification of β-transitions is based on the lifetime of the decay

$$\frac{1}{\tau} = \int P(p_\beta)\,dp_\beta = g^2|M_{fi}|^2 f,$$

which defines f as an integration over all the energy dependent terms in the spectrum and contains all the fundamental constants excluding the coupling constant. Thus for an allowed transition

$$f_0 = (2\pi^3 c^3 \hbar^7)^{-1} \times \int_0^{p_{max}} (E-E_\beta)^2 p_\beta^2 F(Z,E_\beta)\,dp_\beta.$$

The product $f_0 t_{\frac{1}{2}}$, where $t_{\frac{1}{2}}$ is the half-life ($=\tau$ ln 2), is therefore a measure of $(g^2|M_{fi}|^2)^{-1}$ for an allowed transition and is found to vary between 10^3 and 10^6 in magnitude. Transitions having values near 10^3 are known as 'super-allowed' since they correspond to (almost) complete overlap of the nuclear wavefunctions. They are usually Fermi transitions between states of the same isospin multiplet (e.g. β-decay between mirror nuclei), but there exist a few Gamow-Teller transitions in very light nuclei where the overlap is also complete. The remainder of the range corresponds to a diminished overlap. The *same* product $f_0 t$ for a first forbidden transition will be of the form $\{g^2(|M_{if}|^2/R_N^2)\overline{(kR_N)^2}\}^{-1}$, where the bar denotes a suitable average over the β-spectrum. For complete overlap this is up by a factor $(kR_N)^{-2}$ in value on an allowed transition giving a range of order 10^6 - 10^{10}. Similarly, a second forbidden transition will give $f_0 t$ values > 10^{10}. Thus there is no strict division

between the different classes, but for certain ranges of $f_0 t$ values it is possible to deduce the type with a reasonable degree of confidence.

Note that these last considerations should really apply only to β^- transitions since, as has been mentioned in Chapter 1, an alternative decay mode to β^+ is available. This will be discussed briefly in the next section.

ELECTRON CAPTURE

Since atomic nuclei are in general surrounded by atomic electrons an alternative process to β^+-decay can take place, namely,

$$p + \beta^- \rightarrow n + \nu_e \qquad \text{(within the nucleus)}.$$

This process is of greatest importance at low available energies - indeed there is a range of available energy (see Chapter 1) of 0-1·02 MeV over which β^+-decay is impossible and e^--capture only can take place. Also it is expected to be largest for K-electrons (provided the energy balance is not critical), since they have strongest overlap with the nucleus, and for nuclei of large Z, since the electron orbits are more tightly bound, again leading to greater overlap.

Applying Fermi's golden rule to this process, the density-of-states function will be that of the neutrino alone - the nucleus is constrained to recoil in the opposite direction and both particles have fixed energy. The probability of finding the electron at the origin of the coordinate system is now no longer inversely proportional to the volume of the enclosure but to the volume of the K-orbit (for K-capture), so

once again Ω cancels (it has also disappeared from the density-of-states function for the electron which is now unity - a defined state). If E is used, as before, to denote the maximum available energy for the β^+-decay, then the energy available to the neutrino will be $(E + 1{\cdot}02 - B_K)$ MeV, where B_K is the binding energy of the K-electron. There is now no integral to perform since no variation is allowed of the products, and

$$\frac{1}{\tau_K} = \frac{2\pi}{\hbar} g^2 |M_{if}|^2 \frac{1}{\pi} \left(\frac{Z}{a}\right)^3 \frac{(E+1{\cdot}02-B_K)^2}{2\pi^2\hbar^3 c^3} , \qquad (4.6)$$

where a is the (hydrogen) Bohr radius, so that a/Z is the mean radius of the K-electron wavefunction in the parent nucleus, and where the matrix element M_{if} is the same as before, and $1/\tau_K$ is the probability per second of K-capture. Thus the probability of the process increases as Z^3, whilst that for β^+-decay decreases with Z (due to the Coulomb effect); for light nuclei β^+-decay will dominate except for an energy range in E from the negative value $(-1{\cdot}02 + \beta_K)$ MeV to slightly above zero. But for high Z, K-capture is overwhelmingly predominant for β^+-energies commonly encountered.

RELATIVISTIC THEORY AND PARITY VIOLATION

The account of β-decay so far described has been essentially non-relativistic (NR) in its approach, apart from the use of a relativistic equation connecting momentum and energy - a trivial aspect. For relativistic invariance it is essential that time and space be treated on an equal footing, and this is not the case with the Schrödinger wave equation, which is

of second order in spatial derivatives and first order in the time derivative. Putting $E = i\hbar\,\partial/\partial t$, $p_x = (\hbar/i)\,\partial/\partial x$ into the relativistic (R) equation $E^2 = c^2p^2 + m_0^2c^4$ gives the Klein-Gordon equation which is second order in all four derivatives. However, its solutions are found to conform to Bose statistics, e.g. photons, and the equation cannot be used for spin-$\frac{1}{2}$ particles. Dirac looked into the conditions under which this second-order equation factors out into two first-order equations and further examined the solutions of the resulting equation. He found that the solutions were four-component vectors in a space which was closely allied to the two component space of a NR spin-$\frac{1}{2}$ particle. Indeed two of the components are down by a factor v/c on the other two such that in the limit $v \to 0$, the R system reduces to the NR system. In addition the 'small' components have opposite parity to the 'large' components, creating the possibility of parity violation at relativistic energies.

In β-decay, the approximation $v/c \ll 1$ is rarely valid for the electron and never so for the neutrino - even for tritium decay the β-particle has maximum kinetic energy of 18·6 keV corresponding to $v/c \sim 0{\cdot}27$. Thus the product of $\psi_\beta\psi_\nu$, which in the NR limit gives four terms which can be regrouped into a scalar (S=0) and a three-vector (S=1) gives in the R limit sixteen terms which can be regrouped into two scalars, two four-vectors, and a six-component antisymmetric tensor. The two vectors are respectively polar and axial; the former changes sign under the parity operation whilst the latter does not, e.g. $\underline{r}$ and $\underline{p}$ are polar three-vectors whilst $\underline{r} \wedge \underline{p}$ is an axial

three-vector. Of the two scalars one is a pseudo-scalar in that it changes sign under the parity operation; an example of such a scalar is the triple three-vector product $\underline{a}.(\underline{b}\wedge\underline{c})$, where $\underline{a}$, $\underline{b}$, $\underline{c}$ are polar three-vectors. Such a product can be used classically (e.g. to represent a volume) but the sign property is usually ignored.

Thus a full relativistic treatment is much more complicated than the picture presented here and has built into it terms which mix the parity of the relativistic components. It is possible to choose an interaction in such a way as to eliminate the parity-violating component, and this was believed to be the case until specific measurements were made to test parity conservation in β-decay. Such measurements are designed to determine whether pseudo-scalars are necessary in the description of the reaction. Such a scalar product is $\underline{\sigma}.\underline{p}$, where $\underline{\sigma}$ represents the NR spin of the electron and $\underline{p}$ its momentum or $\underline{J}.\underline{p}$, where $\underline{J}$ is the spin of the initial nucleus and the β-momentum $\underline{p}$ is now measured relative to the same axis as $\underline{J}$. Obviously, if parity is conserved then $\langle\underline{\sigma}.\underline{p}\rangle$ and $\langle\underline{J}.\underline{p}\rangle$ must be zero. The first measurements were made on the decay of ^{60}Co, with the initial state polarized by cooling in a magnetic field. It was then found that the number of β-particles directed into the hemisphere with a component along the field was different from that directed into the opposite hemisphere, indicating that $\langle\underline{J}.\underline{p}\rangle$ was not zero. The final conclusion to be drawn from such measurements was surprising, not so much because parity was not conserved but because its violation appeared to be the maximum possible when extrapolated to the limit $v \to c$.

When polarization measurements are ignored as in the simple determination of the spectrum, then the simple predictions of the Fermi theory are largely correct, but, in more detail, separation into 'large' and 'small' terms of the nucleon wave functions is necessary. Part of a first forbidden transition can arise from the 'small' term of an allowed transition, and vice versa, though in the latter case the effect will not be discernible since it is also reduced by the factor $(kR_N)^2$ relative to the main component; in the former case the factor serves to enhance this extra component. Thus, whilst under most circumstances the simple theory copes with allowed transitions, great caution is needed in extending its results to forbidden transitions.

PROBLEMS

4.1. Solve the one-dimensional problem of transmission through the rectangular potential as given in Fig.4.1 (p.91). Show that in the limit of the barrier width approaching zero, the transmission becomes that calculated for a potential step.

4.2. Solve the integral $\int_{r_n}^{r_0} \gamma dr$ (see p.95) for the more realistic potential of Fig.4.1., where the limits of integration are the classical distance of closest approach and the nuclear radius (use the transformation $r = r_0 \cos^2\theta$). Show that, in the limit $E_\alpha \ll$ barrier height, the Gamow factor $\exp(-2\pi Zze^2/4\pi\varepsilon_0\hbar v)$ is the important term in s-wave barrier penetration.

4.3. Show that the non-relativistic formula for the β-spectrum for an allowed transition gives a

mean kinetic energy of the β of $\frac{1}{3}$ the maximum energy, whilst the relativistic formula, for a very high maximum energy, gives a factor $\frac{1}{2}$. Derive expressions for the maximum recoil energy of the nucleus of mass A when the β is emitted at the mean energy in both cases. (Neglect Coulomb effects.)

4.4*. Derive an expression for the electron momentum spectrum in allowed β-decay (ignore Coulomb corrections). Find the approximate dependence of the total decay rate on the maximum electron momentum p_0 when $p_0 \gg mc$.

Compute the value of the partial decay rate for the pion decay $\pi^+ \to \pi^0 + e^+ + \nu_e + 4{\cdot}5$ MeV given that the Fermi decay of $^{14}0$ to ^{14}N (excited) has a momentum end point $p_0 = 2{\cdot}26$ MeV/c and a mean lifetime of 103 s.

4.5*. Explain the meanings of the following terms: (a) first forbidden, allowed and super-allowed β-transitions, and (b) Fermi and Gamow-Teller matrix elements. Classify with regard to (a) and (b) the following: (i) $n \to p$; (ii) $^6_2\mathrm{He}(0^+) \to {}^6_3\mathrm{Li}(1^+)$; (iii) $^{14}_8\mathrm{O}(0^+) \to {}^{14}_7\mathrm{N}(0^+)$, $f_0 t = 3{\cdot}3 \times 10^3$; (iv) $^{35}_{16}\mathrm{S}(\frac{3}{2}^+) \to {}^{35}_{17}\mathrm{Cl}(\frac{3}{2}^+)$, $f_0 t = 1{\cdot}0 \times 10^5$; (v) $^{36}_{17}\mathrm{Cl}(2^-) \to {}^{36}_{18}\mathrm{Ar}(0^+)$.

4.6. $^7_4\mathrm{Be}$ decays by electron capture chiefly to the ground state of $^7_3\mathrm{Li}$, and the recoil energy of the $^7\mathrm{Li}$ has been measured to be 57 eV to 1 per cent accuracy. If the mass difference between the two atoms is 0·862 MeV/c^2 show that the re-

sult is consistent with a neutrino rest mass anywhere in the range of zero to $\sim$ 10 keV/c^2.

5. Spontaneous decay of nuclei II: electromagnetic transitions

Thus far the discussion has been restricted to the decay of nuclear ground states, apart from the occasional intrusion of an excited state decaying, for example, as an isometric state, or giving a 'long-range' α-particle. In the next chapter states at high enough excitation to emit nucleons will be considered, but here the electromagnetic decays of states within 1 MeV or so above the ground state are treated. The electromagnetic process is, in general, a stronger one than the weak β-process, so that such excited states only rarely β-decay, though there are a few instances when the excited states lie close to the ground.

The treatment of radiation presented here is largely based upon the corresponding process of electric dipole radiation from excited atomic states. Later in the chapter it will be shown that such radiation can take place only between states satisfying the criteria $\Delta J = 0, \pm 1$ and change of parity. But, experimentally, using techniques to be described briefly in the next chapter, it is found that in nuclei γ-transitions frequently take place between states not conforming to these criteria. It is therefore necessary to extend the treatment to cover the case of emission via the higher electric multipoles and via magnetic multipoles. Such transitions can be ignored in atoms since they occur so seldomly and then only under very special conditions. If the rapid process of electric dipole (E1) radiation cannot take place,

then the state usually de-excites by collision. In nuclei, on the other hand, it transpires that E1 emission is not a great deal faster than the other processes, and in any case most low-lying states have the same parity as each other since they probably arise from different couplings of the same shell model particles or from collective motion. But nuclei are isolated from each other by virtue of their charge, and are coupled only weakly to the atomic electrons; a state once formed can only decay via the electromagnetic process, though it need not necessarily lead to γ-emission since the Coulomb coupling to the atomic electrons can result in the emission of an electron termed, misleadingly, a 'conversion electron'. This latter process is electromagnetic and gives the same information concerning nuclear states as does γ-emission.

γ-RADIATION

The difficulty in calculating the spontaneous decay of an excited state is that one does not know the perturbing term in the wave equation which is responsible for the transition. However, the reverse process of absorption can be treated by using the electric vector of the radiation as a time-dependent perturbation. Einstein, using a statistical argument (known as 'Einstein's *A*s and *B*s'), obtained a relationship between the two processes. A fuller account is given in Pauling and Wilson (1935), but very briefly the argument is as follows.

The coefficient of absorption is defined such that the probability of a transition from a lower state n to a higher state m is given by $B_{n\to m}\ \rho(\nu_{mn})$,

where $\rho(\nu_{mn})\, d\nu$ is the energy density of radiation in the range ν_{mn} to $\nu_{mn} + d\nu$) with ν_{mn} the frequency of the transition, i.e. $\nu_{mn} = (E_m - E_n)/h$. The probability of transition down is given by $A_{m\to n} + B_{m\to n}\rho(\nu_{mn})$, where $A_{m\to n}$ is the coefficient of spontaneous emission and $B_{m\to n}$ is the coefficient of induced emission. If we consider the equilibrium of this two-state system in a bath of radiation at infinite temperature then $B_{n\to m} = B_{m\to n}$, since the two states will be present in equal proportions, and the induced processes will be much more probable than the spontaneous process. It is assumed that the two states are non-degenerate; this can be achieved by applying a magnetic field to separate out sub-states of n and m - the transitions between a single sub-state in each level is then under consideration. If now the equilibrium is considered at a finite temperature, then the ratio of abundances will be given by the Boltzmann factor, $N_m/N_n = \exp\{-(E_m - E_n)/kT\}$, whilst $\rho(\nu)$ is given by Planck's radiation law,

$$\rho(\nu) = (8\pi h\nu^3/c^3)\{\exp(h\nu/kT)-1\}^{-1} .$$

But at equilibrium

$$N_n/N_m = \{A_{m\to n} + B_{m\to n}\ \rho(\nu_{mn})\}\{B_{n\to m}\ \rho(\nu_{mn})\}^{-1} .$$

From these equations

$$A_{m\to n} = (8\pi h\nu_{mn}^3/c^3)B_{n\to m}$$

$$= (2\hbar\omega_{mn}^3/\pi c^3)B_{n\to m}$$

It is not proposed to go through the complete calculation of $B_{n\to m}$ as given in the above reference for E1 transitions, but to show how to modify this approach in order to produce the other multipoles. The perturbation resulting in absorption was taken to be due to the electric vector of the radiation field, $\underline{E}_0 \exp\{i(\underline{k}.\underline{r}-\omega t)\}$. Since $k = mv/\hbar = E/\hbar c$, for a photon of $\sim$ 1 MeV and a nuclear radius of 5 fm, then $kR_N \sim \frac{1}{40}$. Thus the simplification $E_0 \exp(-i\omega t)$ is a fairly good one, though not so good as in the atomic case. Now a uniform electric field is given by a dipole at infinity, so this approximation, giving a perturbing potential $-e\underline{E}_0.\underline{r} \exp(-i\omega t)$, leads to E1 absorption of amplitude proportional to the matrix element $\int \psi_m^* (\sum_i ez_i)\psi_n d\tau$, where the summation i is taken over the protons of the nucleus and where E_0 has been taken as directed along the z-axis. This gives

$$A_{m\to n} = \left(\frac{4\omega_{mn}^3}{3\hbar c^3}\right)\left|D_{mn}^{(z)}\right|^2 = \left(\frac{4\omega_{mn}^3}{9\hbar c^3}\right)\left|D_{mn}\right|^2, \qquad (5.1)$$

where in the latter expression averaging over all directions of E_0 has been made in forming the dipole matrix element - and also averaging over all sub-states m and n, which is correct for the initial state but not for the final state. A further factor $(2J_n + 1)$ should therefore be present, but it is a degree of refinement which will be ignored. The suitably averaged dipole matrix element becomes e times the radial integral contribution to the moment for a particular choice of sub-states.

The higher electric multipole contributions can be obtained by using the expansion $\exp(i\underline{k}.\underline{r}) = 1 + i\underline{k}.\underline{r} - \frac{1}{2}(\underline{k}.\underline{r})^2$ ..... in the expression

for the radiation field. The leading term has given a component of the E1 matrix element; the next term gives a component of the E2 matrix element $\int\psi_m^* ke(\sum_i x_i z_i)\psi_n d\tau$ if the propagation vector $\underline{k}$ is taken along the x-axis (it must be perpendicular to $\underline{E}_0$, since electromagnetic waves are transverse). And similarly for higher orders.

From the matrix elements the selection rules arise. Since z is an odd function, the product $\psi_m^*\psi_n$ must also be odd to give a non-zero dipole term, i.e. there must be a change of parity. Similarly, xz is even on reflection through the origin, so the E2 transition is characterized by no change of parity. In more detail, if the particle making the transition is in a state of orbital angular momentum $(l_i m_i)$ before and $(l_f m_f)$ after the transition, then the angular dependence of the above component of the E1 matrix element will be $\int Y^*_{l_f m_f}(\theta,\phi)\cos\theta\ Y_{l_i m_i}(\theta,\phi)d\Omega$, and the properties of the spherical harmonics (Pauling and Wilson 1935) are such that this integral is zero unless $m_f = m_i$ and $l_f - l_i = \pm 1$ (there are two other components which lead to $m_f - m_i = \pm 1$). Similarly the above component of the E2 matrix element, $\int Y^*_{l_f m_f}(\theta,\phi)\cos\theta\sin\theta\cos\phi\, Y_{l_i m_i}(\theta,\phi)d\Omega$, will be zero unless $m_f - m_i = \pm 1$ and $l_f - l_i = 0, \pm 2$, with $0 \to 0$ excluded since it cannot satisfy the former condition. Again, there are other components, for which $m_f - m_i = 0, \pm 2$, e.g. the xy component giving $\Delta m = \pm 2$.

For nuclei consisting of a single proton outside closed shells, the selection rules on (J,M) will be the same as those obtained on (l,m) for transitions between states corresponding to excitations of this

proton, apart from the addition of a spin vector. In the general case it is necessary to decouple the rest of the nucleus from the particle making the transition and then to recouple after the transition. The mathematics of these coupling operations is beyond the scope of this book, but it is not difficult to see that the selection rules will become

E1 $\Delta J = 0, \pm 1$ ($0 \not\to 0$ excluded), with parity change,

E2 $\Delta J = 0, \pm 1, \pm 2$ ($0 \not\to 0, \frac{1}{2} \not\to \frac{1}{2}$ excluded), with no parity change.

The exclusions arise because all components of a given moment must be allowed since they transform into each other on rotation of axes. Thus $\Delta m = 2$ (and therefore $\Delta M = 2$) must be possible in E2, and it is patently not so for the excluded transitions.

So far no mention has been made of magnetic multipole transitions for the very good reason that the interaction has been taken to be that of an electric field. But electromagnetic radiation also possesses a magnetic vector $\underline{H}_0$, perpendicular to both $\underline{E}_0$ and $\underline{k}$. Again the leading term will arise by assuming the (magnetic) field constant over the nucleus giving rise to the perturbation term $-\underline{\mu}.\underline{B}_0 \exp(-i\omega t)$, where the magnetic moment $\underline{\mu} = \beta_N \sum_i (g_s^i \underline{s}_i + g_l^i \underline{l}_i)$, where β_N is the nuclear magneton and g_s^i, g_l^i are the appropriate spin, orbital g-factors for the nucleon i. To simplify matters, consider the case of a single neutron outside a closed-shell core then $\underline{\mu}.\underline{B}_0 = \hbar^{-1} g_s \beta_N (s_x B_{0x} + s_y B_{0y} + s_z B_{0z})$, where s_x, s_y, s_z are spin operators (see Appendix D). If the nucleus

conforms to the J-J coupling scheme (and most do, at least approximately, and especially near closed shells) then the two states $J = l \pm \frac{1}{2}$ are distinct and orthogonal. They can be expressed in terms of orbital and spin product wavefunctions thus:

$$|l+\tfrac{1}{2},m\rangle = \alpha|l,m+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + \beta|l,m-\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle,$$

$$|l-\tfrac{1}{2},m\rangle = -\beta|l,m+\tfrac{1}{2}\rangle\ \tfrac{1}{2},-\tfrac{1}{2}\rangle + \alpha|l,m-\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle,$$

where α and β are normalized such that $|\alpha|^2+|\beta|^2 = 1$ (they are known, but the values are not needed here). Since $s_z = \frac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ it follows that $s_z|l+\frac{1}{2},m\rangle$ contains both $|l+\frac{1}{2},m\rangle$ and $|l-\frac{1}{2},m\rangle$ as also does $s_z|l-\frac{1}{2},m\rangle$. In treating s_x and s_y it is simpler to recombine them to form $s^+ = s_x+is_y$ and $s^- = s_x-is_y$ with weighting factors which will depend upon B_{0x} and B_{0y}. Now, from Appendix D, $s^+|l+\frac{1}{2},m\rangle \propto |l,m+\frac{1}{2}\rangle|\frac{1}{2},+\frac{1}{2}\rangle$, which is a linear combination of $|l+\frac{1}{2},m+1\rangle$ and $|l-\frac{1}{2},m+1\rangle$; and similarly for s^-. Thus the effect of $\underline{s}.\underline{B}_0$ is to mix the original state $|j,m\rangle$ into states belonging to both couplings of $\underline{l}$ and $\underline{s}$ and also to mix the orientations, $\Delta m = 0, \pm 1$. Thus the matrix element will be non-zero if the final state contains one or other of these components. A similar argument applies for the term in $\underline{l}.\underline{B}_0$ for the proton. The selection rule for magnetic dipole single-particle transitions is

M1 $\Delta l = 0$, $\Delta m = 0, \pm 1$, $|\Delta j| = 0, 1$, no change of parity,

where $|\Delta j| = 1$ allows the alternative coupling scheme. In the general case it is necessary to decouple and

recouple with the rest of the nucleus giving

M1 : $\Delta J = 0, \pm 1$, no change of parity, ($0 \to 0$ excluded).

The next step would be to expand the exponential again to give higher magnetic multipoles, but we shall leave it at this point, except for observing that the parity will alternate with the higher magnetic multipoles, as for the electric. It is interesting to note that, whilst E1 transitions are essentially from one shell to another, M1 and E2 transitions can be made between states with the same basic shell structure. In fact most low-lying nuclear states are different configurations of the same basic states so one encounters M1 and E2 transitions much more often than E1.

From the foregoing it is possible to make crude estimates of the ratios of these three main types of transition. E2 is seen to be down by a factor $(kR_N)^2 \sim 1/2000$ on E1 for a medium-weight nucleus, whilst

$$(\mathrm{M1/E1})^{1/2} \sim \hbar^{-1}\beta_N g_s B_0 \int \psi_m^* s_z \psi_n \, d\tau / eE_0 \int \psi_f^* z \psi_i \, d\tau$$

$$\sim (\beta_N g_s / 2eR_N)(B_0/E_0)$$

$$= (\hbar / McR_N)(g_s/4) \ .$$

From the uncertainty principle $\hbar/R_N \sim Mv$ leading to a ratio of intensities $\sim v^2/c^2$ which, for a typical kinetic energy inside the nucleus of order 50 MeV, takes the value $\sim \frac{1}{10}$. This is perhaps an overestimate, a factor $\sim \frac{1}{100}$ is closer.

Blatt and Weisskopf (1952) make estimates of the

matrix elements under simple assumptions arriving at $\frac{3}{4}R_N$ for E1 and $\frac{3}{5}R_N^2$ for E2. For M1 they take $M1/E1 \sim 10(\hbar/McR_N)^2$, where the factor 10 allows for the large-spin g-factors of proton and neutron. Inserting these estimates and the values of the physical constants gives

$$\Gamma_\gamma(E1) = 0{\cdot}07E_\gamma^3 A^{2/3}$$

$$\Gamma_\gamma(E2) = 4{\cdot}9 \times 10^{-8}\, E_\gamma^5 A^{4/3},$$

$$\Gamma_\gamma(M1) = 0{\cdot}021E_\gamma^3 , \qquad (5.2)$$

where Γ_γ is in eV, E_γ in MeV, and A is the mass number. The radiation width Γ_γ is connected with the decay half-life by $t_{1/2}(s) \sim 4{\cdot}6 \times 10^{-16}/\Gamma_\gamma$ (see also Fig. 5.1).

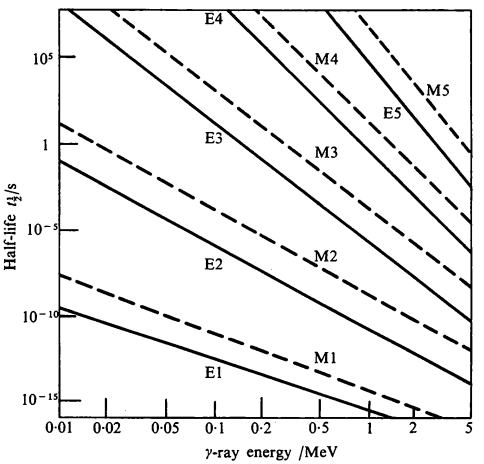

Fig.5.1. Lifetime-energy relationships for γ-radiation, based on Weisskopf's single-particle estimates.

Thus it is seen that the single-particle estimates lead one to expect E1 transitions to be stronger than M1 or E2 at $\sim$ 1 MeV. But, as remarked previously transitions between single-particle states will rarely be found at such low excitation. In addition, there appears to be a tendency for neutrons and protons to move together in nuclei, and, from the remarks below concerning centre-of-mass effects, this will very much reduce the effective charge of the radiating system for E1 transitions. Except for a few transitions in the light elements low-energy E1 transitions are many orders of magnitude weaker than single particle. Since M1 transitions are expected between similar components in complex wavefunctions, they are rarely found to approach single-particle strength, and are typically a few per cent of this value. On the other hand, E2 transitions are often found to be a factor of 100 stronger than the single-particle estimate. The matrix elements so far discussed have been treated implicitly in terms of the shell model in which only a small number ($\sim$ 1) of particles can find suitable initial and final configurations. If, however, all Z protons move coherently, then it is possible in principle to obtain transitions up to Z^2 stronger than single particle. In a rotational band the states have such a correlated motion in which at least 20 nucleons are taking part.

The switch from protons to nucleons is deliberate; it is found that neutron transitions are as strong as proton transitions for both E1 and E2, though for different reasons. In the former case it is because of the finite mass of the nucleus; when a proton moves outward, the rest of the nucleus must move in the

opposite direction to keep the centre of mass unchanged. By simply taking first moments (in the classical sense) about the centre of mass it is seen that the system behaves as though it were infinitely massive provided that the charge on the proton is taken as $+e(1 - Z/A) \sim + \frac{1}{2}e$. A similar argument gives the effective charge on the neutron as $-eZ/A \sim -\frac{1}{2}e$. Since first moments were taken this applies to E1 transitions. Performing the same operations with second moments gives only small departures from $+e$ and zero for the two nucleons. Thus the centre-of-mass effect does not produce large E2 neutron transitions. That E2 neutron transitions are as large as proton transitions even in nuclei like ^{17}O and ^{17}F (closed shell plus one neutron or proton) must be ascribed to a collective polarization of the core, and in both cases it must be a large effect.

Since E2 are in general larger than single particle and M1 in general smaller, they are found to be comparable in strength. The commonest transitions in nuclei are therefore found to be M1 or E2 or mixed M1/E2 transitions; in the latter case either type can be larger though with a tendency towards E2 in the heavier, distorted nuclei and to M1 in the lighter nuclei or nuclei near to closed shells.

INTERNAL CONVERSION

It is possible for an excited nucleus in an atom to de-excite by emitting an atomic electron by an electromagnetic process known as internal conversion. It has an atomic counterpart known as the Auger effect when, as an alternative to X-radiation, an outer-shell electron is emitted instead. This process could be

looked upon merely as the photoelectric effect in which the atom happens to be the same one as contains the radiating nucleus; indeed, such a process must occur and is more likely to occur for this particular atom than any other atom, since the radiation density is vastly greater than for any other atom. However, the photoelectric effect is not very probable for 'any other atom', so even for the particular atom it is likely to account only for a very small fraction of decays. But internal conversion can account for the major fraction of low-energy transitions in heavy nuclei, so a single-stage process is indicated in which the energy is transferred directly from the nucleus to the ejected electron. The Coulomb field provides the perturbation for the process.

The Coulomb potential at a given point $\underline{r}_e$ in space due to the nucleus is of the form

$$V(\underline{r}_e) = \sum_i \frac{e}{|\underline{r}_e - \underline{r}_i|} \quad \frac{1}{4\pi\epsilon_0} ,$$

where the summation is taken over the protons, of position vector $\underline{r}_i$, inside the nucleus. If the field point $\underline{r}_e$ lies outside the nucleus, i.e. $|\underline{r}_e| > |\underline{r}_i|$ for any $\underline{r}_i$ inside the nucleus, then the term $1/|\underline{r}_e - \underline{r}_i|$ can be expanded in powers of $\underline{r}_i . \underline{r}_e / r_e^2$. When r_i is averaged over the nucleus each term in the expansion produces a nuclear electric multipole moment just as in the corresponding expansion of the radiation field. There is also a similar expansion in terms of magnetic multipoles arising from the magnetic interaction between the nuclear magnetic moment and that of the electron. Thus γ-emission and internal conversion depend upon the same nuclear property. Without going into details,

the process is expected to be more probable for the heavier nuclei, since the electron being ejected is then on average, closer to the nucleus and subject to a larger electromagnetic field. Also from considerations of density of final states, the process is expected to be larger, relative to γ-emission, for lower transition energies, provided they exceed the binding energy of the electron. The heavier the particle the greater its momentum for a given energy (assuming that energy is not required to create the particle) and photons have zero rest mass. The ratio of densities of states favours internal conversion most strongly just above threshold and least strongly at high energies. The details of the process also favour conversion just above threshold.

The internal conversion coefficient α is defined as the ratio of probabilities of the conversion process to γ-emission. It can be further divided into components $\alpha = \alpha_K + \alpha_L + \alpha_M$, etc. corresponding to the shell from which the electron is emitted. Its behaviour is shown graphically in Fig.5.2. The coefficients α_K, α_L, α_M, etc. have been extensively tabulated for all atoms over a wide range of energies and multipolarities. Measurements of α, α_K, α_L, α_M are used to determine the type of transition and even the amount of mixing in a mixed transition. Since the process is one of two-body break-up, the energy spectrum consists of discrete lines separated by the binding-energy differences of the atomic shells. Such lines were first discovered when superimposed on continuous β-spectra and initially led to confusion in the interpretation of β-spectra.

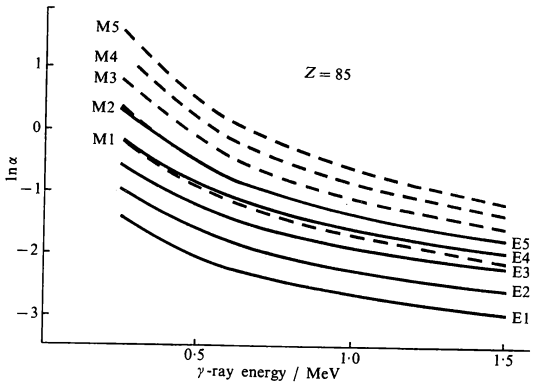

Fig.5.2. Internal conversion coefficients (α) as functions of energy and multipolarity for a heavy element. (Based on SEGRE (1959). *Experimental nuclear physics*, VOL.III, Wiley.)

$J = 0 \rightarrow 0$ TRANSITIONS

It is not strictly correct to say that the conversion process has exactly the same nuclear dependence as that of γ-emission. The statement arose because of the similarity of two expansions and the expansion of $1/|\underline{r}_e - \underline{r}_i|$ is of the stated form only if $\underline{r}_e$ lies outside the nucleus. Whilst an atomic electron is usually well outside the nucleus, it does overlap it for a small fraction of the time; when $|\underline{r}_e| < |\underline{r}_i|$ the expansion becomes quite different, and in particular the monopole component, which for an external point gives a time-independent field because centre of charge and centre of mass coincide, can now have a time variation. Looked at simply, a monopole oscillation of the nucleus is a pulsating mode, having no effect on an external point, but for an internal point the potential depends upon the instantaneous nuclear radius and this is oscillating in time.

Although the finite size effect is small for transitions capable of radiating, for $0 \rightarrow 0$ transitions it is crucial. There are examples of such transitions of less than 1 MeV which can de-excite by no other process - the simultaneous emission of two γ-rays is possible but has never been detected: An example is the first excited state of ^{72}Ge (0^+, 690 keV) which decays in this way with a half-life of 0·3 μs. Above 1 MeV the process in the next section becomes possible.

INTERNAL PAIR CREATION

In this process a state of excitation greater than $2\, m_e c^2$ (1·02) MeV) can de-excite by the creation and emission of an $e^+ - e^-$ pair. Since the emission can take place from inside the nucleus, $0 \rightarrow 0$ transitions can occur this way. Also the density of final states will increase more rapidly with energy for this three-body break-up than for internal conversion, so at high enough energies this process will dominate in $0 \rightarrow 0$ transitions. An example is the first excited state of ^{16}O (0^+, 6 MeV) which decays by pair-emission in 5×10^{-11}s. Pair-emission in competition with γ-emission has also been observed. Again in principle it should dominate at high energies, from phase-space arguments, but long before it does so the levels have become virtual with many channels open for the emission of nucleons.

PROBLEMS

5.1. Estimate the lifetime of the first excited state of a (spinless) proton in a cubic box of side 5 fm. (Consult Pauling and Wilson (1935) for

the quantum-mechanical problem.)

5.2*. What is meant by multipolarity of electromagnetic radiation? Explain why multipolarities other than E1 are commonly observed in nuclei but only rarely in atoms.

The decay scheme of low-lying levels in ^{14}N is (A; 0; 1^+, 0; -), (B; 2·3; 0^+, 1; A, 100 per cent), (E; 5·1; 2^-, 0; A, 67 per cent; B, 33 per cent), (C; 3·9; 1^+, 0; B, 96 per cent; A, 4 per cent) (D; 4·9; 0^-, 0; A 100 per cent), (E; 5·1); 2^-, 0; A, 67 per cent; B, 33 per cent), where the capital letter designates a level, of energy in MeV given next, followed by J^π, T, and finally the γ-decay of the level in percentages to the levels below. Sketch the level scheme and the transitions. Explain the significance of J^π, T and give the possible and expected multipolarities of transitions. Comment on the absence of the transitions DC, DB, ED, and EC. What other decay processes can occur?

5.3. $^{160}_{65}Tb$ decays by β-emission to states of $^{160}_{66}Dy$. Two of the γ-rays resulting are associated with an internal-conversion line spectrum as follows: 32·9, 78·0, 84·7, 86·1, 143·0, 188·6, 195·1, 196·5 keV. The K,L.M,N, edges in Dy are 53·8, 8·6, 1·9, and 0·4 keV respectively. Find the γ-energies, identifying each conversion line with the nuclear transition and the atomic shell from which it arises. (Ignore the splitting of the L,M,N levels.)

5.4. Determine the effective charges due to centre-of-mass effects for neutron and proton E1 and E2

transitions in $N = Z$ nuclei, using the argument outlined on p.127.

5.5. From the result of Problem 5.4, expressing the dipole operator in $N = Z$ nuclei as $-e \sum_i \tau_z^{(i)} \underline{r}_i$, show that no E1 transition is possible between $T = 0$ states. (Consider specifically the isospin component of the wavefunction of a proton and a neutron outside closed shells.) Return to Problem 5.2 to determine whether this new selection rule has relevance.

6. Nuclear reactions

INTRODUCTION

In the earlier chapters the decay of unstable ground states was considered, and in the previous chapter the decay of low-lying excited states. The scope is extended here to states of high enough excitation to decay by particle-emission and also to consideration in more detail of the setting up of the low-lying states which decay by γ-emission. The experimental technique is to bombard a suitable material (the target) with a beam of nuclear projectiles and to observe the reaction products, or to select them for observation using coincidence techniques. If the projectiles are charged particles (p, d, α, ^{12}C, ^{16}O, etc.) they can be produced in quantity in a well-focused beam of well-defined energy from an accelerator, e.g. a cyclotron or an electrostatic generator. If the projectiles are neutrons, then they must first be produced by a nuclear reaction, selected in energy as well as possible, and collimated (rather than focused) onto the target; the products from the second nuclear reaction are then observed - against the background produced by neutrons from the first reaction emitted in directions other than that of the second target. A copious source of neutrons is a reactor (see Chapter 7), especially for neutrons of low energy (the keV and eV region); but for neutrons of energy a few hundred keV and greater it is preferable to produce them using a primary reaction induced by charged particles from an accelerator.

The study of nuclear reactions has three basic aims:

(1) to locate levels in a nucleus;
(2) to determine nuclear properties of levels;
(3) to study reaction mechanisms, i.e. nuclear dynamics as distinct from the (quasi-) static behaviour in (2).

Experimentally there is much overlap of the three aims, especially (1) and (2), since the location of levels and decay mechanisms can often be determined in the same experiment. To locate levels we need to be able to handle the kinematics of nuclear reactions; to determine their properties we need to look at the decay modes and measure their lifetimes. For (3), it is intended to limit the discussion to a brief description of the direct and compound nucleus reaction mechanisms.

THE KINEMATICS OF NUCLEAR REACTIONS

A nuclear reaction can be represented as

$$A + a \rightarrow B + b + Q,$$

where a is the incident projectile and A the target nucleus, at rest in the laboratory frame; b is a light particle emitted in the reaction and B the final nucleus (the distinction between light and heavy final product is one of convenience; it is not always possible to make it - an example being fission (see Chapter 7)). The 'heat of reaction' Q, as defined above, is positive if the kinetic energies of the final products exceed that of the initiating projectile. The extra energy comes from the internal poten-

tial energies of the particles, and these are reflected in their masses. Elastic scattering is a particularly simple example of a nuclear reaction, having $Q = 0$. In general, the final nucleus need not be in its ground state; an increased state of excitation results in a reduced Q-value. From the kinematics of the reaction the Q-value is determined and so the state of excitation of B, provided the ground-state transition can be recognized.

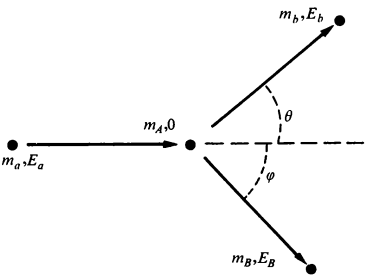

Fig.6.1. Schematic representation of the kinetics of the reaction A + a → B + b.

If the reaction is represented in the laboratory frame as in Fig.6.1, then using non-relativistic mechanics (accurate enough for most reactions using electrostatic generators and small cyclotrons):

conservation of momentum:

$$\sqrt{(2m_a E_a)} = \sqrt{(2m_b E_b)}\cos\theta + \sqrt{(2m_B E_B)}\cos\phi$$

$$0 = \sqrt{(2m_b E_b)}\sin\theta - \sqrt{(2m_B E_B)}\sin\phi ,$$

which on re-ordering, squaring, and adding give

$$m_B m_B = m_a E_a + m_b E_b - 2\sqrt{(m_a m_b E_a E_b)}\cos\theta ,$$

conservation of (kinetic) energy:

$$E_a + Q = E_b + E_B,$$

giving

$$(m_B+m_b)E_b - (m_B-m_a)E_a - 2\sqrt{(m_a m_b E_a E_b)}\cos\theta - m_B Q = 0, \tag{6.1}$$

conservation of (mass) energy:

$$m_A + m_a = m_B + m_b + Q/c^2 . \tag{6.2}$$

Note that in the non-relativistic limit it is convenient to express the relativistic conservation of mass-energy as two separate equations connected by the heat of reaction, for a reason which becomes apparent in the next paragraph.

In the general experiment, we measure the energy of emission of b at a given angle of emission θ relative to the beam. In equations (6.1) and (6.2) it is assumed that m_a, m_A, m_b, E_b, and θ are known (given data or by measurement) and m_B and Q are unknown. Thus (6.1) and (6.2) can be used to determine m_B and Q. In point of fact it is rarely necessary to solve these cumbersome simultaneous equations in the normal way. From examination of the mass values of the stable and nearly stable nuclei it will be seen that the error incurred by using the mass number A for the mass of an unknown nucleus (and A will be known by

conservation of nucleons) in (6.1) is probably of the order of 0·1 per cent, and certainly not greater than 0·3 per cent for all but the very lightest nuclei. It is therefore possible to solve (6.1) directly for Q to an accuracy $\sim$ 0·2 per cent; since in (6.2) the term in Q is very small, the error incurred in deducing an accurate mass value m_B from (6.2) using this inaccurate Q is probably less than the errors arising from the quoted mass values of the other particles. In any case, if it is justified to work to greater accuracy relativistic effects should be considered.

If we have measured a number of different energies of particle b at a given θ, corresponding to different energy levels in the final nucleus B (assuming that b itself has no excited state - true for n, p, d, α), then to each energy E_i can be assigned a Q-value Q_i, where Q_0-Q_i is the energy of the level in nucleus B. Q_0, the Q-value of the transition to ground, is either known or recognizable from a previously known pattern of excited states. Often the largest Q-value is assumed to arise from the transition to ground - occasionally an inaccurate assumption.

Often equation (6.1) is used in reverse. In a given spectrum we may suspect that there may be spurious lines due to impurities in the target. We therefore wish to determine energies corresponding to levels in the final nucleus following reaction with the target impurity, for which Q_0 is known and the energy levels of the final nucleus. Thus in (6.1) all is known except E_b and this must be solved either as a quadratic in $(E_b)^{\frac{1}{2}}$ or by iteration.

Finally, in measuring differential cross-sections,

it must be appreciated that the correct coordinate system to use is the centre-of-mass frame, not the laboratory frame. This affects the measurements in that the transformation from the latter to the former frame changes both the angle and the effective solid angle of the detector. The calculation has been left as an exercise.

PROPERTIES OF LEVELS

Having located (bound) levels in a nucleus using the above kinematic analysis, it is desired to determine other characteristics of these levels such as spin, parity, γ-decay spectrum to lower-lying levels, half-life for γ-decay, magnetic moment, electric quadrupole moment, and less obvious characteristics such as the extent to which a particular level looks like a single particle (proton or neutron) outside a core which could be the ground state (or an excited state) of the appropriate neighbour.

A great deal of information can be obtained by detecting the γ-radiation from the state, either directly or in coincidence with the particle which resulted in the formation of the state. The relative intensities of γ-transitions to known levels can give useful information on application of the selection rules. By the method of 'delayed coincidences' it is possible to plot the decay curve of the state and so determine its half-life. Used in conjunction with Weisskopf estimates (see p.125), it may then be possible to determine the spin and parity of the state. A useful technique is to measure the angular correlation of the γ-radiation either relative to the beam, with the outgoing particle detected in a certain direc-

tion (a most convenient direction is along, or against, the beam, since the problem then has axial symmetry).

In order to see how information can be obtained from the direction of emission of particles from the nucleus, consider an idealized nuclear state which looks like an α-particle in an $l = 1$ orbit outside a closed shell core. Then, relative to a given axis, the angular components of the wavefunction will be of the form $Y_{1,m}(\theta,\phi)$. If the reaction setting up the state was selective in m and ensured that it was in the $m = 0$ sub-state then the angular wavefunction is $Y_{1,0}(\theta,\phi) \propto \cos\theta$; if also this state is unstable to the emission of α-particles, then the wavefunction of the free α-particle must have the same angular dependence (a necessary condition for matching wavefunctions at a spherical boundary). So the probability of observing an α-particle at an angle θ to the given axis will in this simple case be proportional to $\cos^2\theta$. It is unfortunate that the example chosen is that of an unbound state when discussing the γ-decay of particle-stable energy levels, but the object was to avoid the additional complexities of spin when considering effects of orbital angular momentum and this is not possible for γ-transitions (nor nucleon emission). Angular dependence is, however, implicit in the discussion of γ-emission in the previous chapter, where it was indicated that γ-absorption by a $(J=0,M=0)$ orbital state of a proton to form a (1,0) orbital state arose from the z-component of the radiation field. Since the field is perpendicular to the propagation vector, the z-component will contain the factor $\sin\theta$, where θ is the angle between the propagation vector and the z-axis. Thus the probability

of absorption $\propto \sin^2\theta$ and, by time-reversal, the probability of decay of a (1,0) orbital state to a (0,0) orbital state by emission of a photon in a direction θ to the axis of quantization $\propto \sin^2\theta$. Another way of presenting this is merely to state the classical result that a dipole along the z-axis emits a radiation field whose amplitude is proportional to $\sin\theta$.

Thus, in general, if a level can be excited in a single sub-state relative to a given axis then the decay products (whether particle or γ-ray) will be emitted with an angular variation relative to that axis. A corollary to this statement is that if the level has equal population of all sub-states then the radiation will be isotropic, because lack of knowledge concerning the formation of the state can only be expressed in this way. It is rarely possible to set up a decaying level as a single sub-state, but neither is it necessary to do so to observe an angular dependence of the decay; a non-random population of the sub-states is sufficient, and this occurs whenever the particle initiating the reaction is absorbed out of a directional beam. Classically, the absorption of a particle moving in a certain direction cannot change the component of orbital angular momentum of the system along that direction and this holds good in making the transition to quantum mechanics. It is therefore convenient to make the beam direction the axis of quantization; the setting up of the state by absorption of a particle out of the beam is then characterized by $\Delta m = 0$. Thus if an α-particle were to be absorbed by a 0^+ target to form a 1^- state then this state could only be formed as the (1,0) sub-state, and its subsequent decay by γ-emission to a 0^+ ground

state will give an angular distribution $\propto \sin^2\theta$. If the target ground state has angular momentum J and the α-absorption produces a state $J' = J + 1$ then the sub-state (J,M) will result in the formation of $(J + 1,M)$. Since nothing is known about the target condition the angular distribution arising from each initial (J,M) must be summed with equal weighting. But in the intermediate state, the sub-states $(J + 1, J + 1)$ and $(J + 1, -J - 1)$ cannot be formed at all, so the averaging in the initial state is not equivalent to random averaging in the intermediate state and is not likely to produce isotropy. To proceed further requires a detailed study of the mathematics of coupling of angular momentum in quantized systems; here it suffices to state that an angular distribution can be calculated in terms of a few nuclear parameters, the J^π of the states involved, the mixing of emitted radiations (e.g. M1 and E2), etc., and checked against measurement to give information concerning these parameters.

Two important qualitative features can be appreciated. If the intermediate state has $J = 0$ then any radiation subsequently emitted must be isotropic - since randomness is represented by equal populations of all sub-states, $J = 0$ must always be in a random condition. Less obviously, if the particle absorbed has $l = 0$, then all subsequent radiations will be isotropic - this follows from the realization that adding nothing (angular-momentum-wise) to randomness can hardly produce other than randomness.

The above outlines the theme, but the variations are numerous and can be complicated. As an example of complication, if a reaction is observed in the presence of a magnetic field then, from Larmor's theorem, the

whole system can be assumed to rotate with the Larmor precession frequency; measuring the angular distribution as a function of time after formation of the intermediate state should reveal a rotation of the angular distribution by an amount which depends upon the magnetic moment of the intermediate state and the time delay and so gives the magnetic moment of this excited state.

The methods for determining to what extent a given state looks like a shell-model state are best left until the next section.

INTERACTION MECHANISMS

Broadly speaking interactions can be divided into two categories, but the division is not sharp. The category of 'direct reactions' refers to reactions which take place whilst the bombarding particle is traversing the target nucleus, i.e. in a time $\sim 2R_N/v$, where v is the velocity of the particle (typically $> c/10$). This characteristic time is 10^{-21} - 10^{-22}s. An example is the reaction A(d,p)B, in which the deuteron, whilst passing through the periphery of nucleus A loses a neutron to form B. In the second category, 'compound-nucleus reactions', the time of interaction is very long compared with the characteristic time above. Thus there is a long time interval between the absorption of the bombarding particle and the emission of the product particle. It is therefore reasonable to talk of the compound nucleus as an intermediate product in the reaction; if the reaction is A(X,Y)B then it effectively proceeds in two steps $A+X \rightarrow C^* \rightarrow B + Y$. Since C* exists for a time long on the nuclear time scale it must correspond to a (quasi-) stationary state of the nucleus C and is in fact just

one of the many virtual states of excitation of C.

As is usual, we define the processes in terms of extremes. The real situation is not always so clear. If an incoming projectile interacts with a nucleon near the surface of the nucleus, it or the nucleon could be scattered clear of the nucleus giving a fast direct reaction; if it gives up some energy to the nucleon but insufficient to raise the latter out of the nucleus and thereby moves further into the nucleus, it can once again be scattered out by another nucleon or share energy with it and still remain inside the nucleus. At each collision it becomes harder for the particles concerned to escape, and finally the energy is well shared among all the nucleons. At this stage the compound-nuclear state has indeed been formed. It can only decay when, by chance, one nucleon can acquire a sufficiently large share of the excess energy to become unbound, and this can take a long time when the energy has been shared over a large number of nucleons. Thus virtual nuclear states tend to be longer lived (for a given excitation) in heavier nuclei. At low energies, and especially for charged particles, the compound-nucleus process will dominate in the reaction cross-section (the elastic Rutherford scattering is large and direct) since, having penetrated the barrier (an improbable process), the particle even if it fails to collide has little chance of penetrating once again at the first pass. It is therefore reflected back and forth at the surface until finally it shares energy and forms the compound nucleus. At high energies direct reactions become of increasing importance - at very high energies it is virtually certain that

both incoming and struck particle will escape from the nucleus.

The compound-nucleus process

The reaction can be described mathematically as arising from a first-order time-dependent perturbation setting up the compound nucleus C* which subsequently decays, or alternatively as arising from a second-order time-dependent perturbation between the initial and final states which is dominated by one channel proceeding through C*. The two processes are in reality the same thing, but it is instructive to look at both.

As always in time-dependent perturbation, we consider the amplitudes a,b,c of normalized wavefunctions of states A,B, and C* to be functions of time. The wavefunction representing C* is taken to be of the form $c_0\exp(-t/2\tau)\psi_C(r,\theta,\phi)$ in order to represent the fact that, left alone, C* will decay in intensity with mean life τ. First-order perturbation theory gives the rate of formation of C* as

$$-(i/\hbar)H_{CA}\exp\{i(E_C-E_A)t/\hbar\}a,$$

where H_{CA} will be assumed to be independent of time, except for having been switched on at $t = 0$. The amplitude c therefore satisfies the differential equation

$$\frac{dc}{dt} = -(1/2\tau)c - (i/\hbar)H_{CA}\exp\{i(E_C-E_A)t/\hbar\}a .$$

If the probability of transition is small, then there is little absorption of the wave representing the initial state, so a = constant = 1 is a reasonable

approximation, and the equation integrates to

$$c = \frac{-(i/\hbar)H_{CA}[\exp\{i(E_C-E_A)t/\hbar\}-\exp(-t/2\tau)]}{(i/\hbar)(E_C-E_A) + 1/2\tau}, \quad (6.3)$$

satisfying the condition $c = 0$ at $t = 0$. Whilst τ is large on a nuclear time scale it is small on a laboratory scale ($\sim 10^{-14}$s or less), and so the term $\exp(-t/2\tau)$ can be neglected under experimental conditions giving

$$|c|^2 = \frac{|H_{CA}|^2}{\{(E_C-E_A)^2+\hbar^2/4\tau^2\}} = \frac{|H_{CA}|^2}{\{(E_C-E_A)^2+\Gamma^2/4\}}, \quad (6.4)$$

where $\Gamma\tau = \hbar$. Now the probability of decay of a state is the sum of its decay probabilities to the various decay channels:

$$1/\tau = \sum_i (1/\tau_i) \quad \text{or} \quad \Gamma = \sum_i \Gamma_i .$$

$$\text{Rate of formation of state B} = \frac{|c|^2}{\tau_\beta} = \frac{|H_{CA}|^2\Gamma_\beta}{\hbar\{(E_C-E_A)^2+\Gamma^2/4\}}.$$

Considered as a second-order perturbation the rate of production of B is

$$(2\pi/\hbar)\left|\sum_c H_{CA}H_{BC}/(E_A-E_C)\right|^2 \times (\text{energy density of final states}).$$

The time-dependence of the (decaying) state C* is

$$\exp(-iE_Ct/\hbar)\exp(-t/2\tau) = \exp\{-i(E_C-i\Gamma/2)t/\hbar\}.$$

Thus the decaying state C* is effectively at an energy $(E_C - i\Gamma/2)$. If in the above summation one term

dominates and that one is C*, then the expression inside the moduli becomes

$$\frac{|H_{CA}|^2 |H_{BC}|^2}{\{(E_A - E_C)^2 + \Gamma^2/4\}}$$

Comparison of the two expressions for rate of formation then gives

$$\Gamma_B = 2\pi |H_{BC}|^2 \times \text{(energy density of final states)},$$

and, by applying a time-reversal argument,

$$\Gamma_A = 2\pi |H_{CA}|^2 \times \text{(energy density of final states of time-reversed system)}.$$

From Chapter 4 (p.101) this is

$$\Gamma_A = 2\pi |H_{CA}|^2 \left\{\frac{4\pi p_a^2}{v_a (2\pi\hbar)^3}\right\} (2J_a+1)(2J_A+1),$$

where the surrounding box has been taken to have unit volume (in any case its size drops out), and the statistical weights of the states have been included. Thus

$$\Gamma_A = (1/\pi\hbar^3) |H_{CA}|^2 m_a^2 v_a (2J_a+1)(2J_A+1) , \qquad (6.5)$$

and the rate of formation of B is then

$$\frac{\pi\hbar^2 (2J_C+1)}{m_a^2 v_a (2J_a+1)(2J_A+1)} \cdot \frac{\Gamma_A \Gamma_B}{(E_A - E_C)^2 + \Gamma 2/4}$$

where the statistical weight of C* has been inserted since the calculation has previously referred to a

single intermediate state. The statistical weights have been introduced rather too glibly. The factor $1/(2J_a+1)(2J_A+1)$ is perhaps obvious since the Γs have been defined in terms of a summation over final states, whereas the initial state has unit normalization spread over the sub-states, i.e. an average over initial states is required. In any case the argument presented is sound; the density of final states in the time-reversed system certainly contains the factor, but the yield does not and therefore must contain $\Gamma_A/(2J_a+1)(2J_A+1)$. At first sight it may appear that, at least in the expression given for the second-order transition, a term like $(2J_C + 1)^2$ arises, since the summation over sub-states is done before taking the square of the modulus. However, the expressions derived are for rates integrated over all directions of emission of the final particle and all cross-products integrate to zero, leaving only the terms which arise if the summation over sub-states were taken after forming the square of the modulus. The term $(2J_C+1)$ is the number of 'final' states produced by the first-order process; that these states decay is not relevant to the first-order treatment. Hence the right side of equantion (6.4) should be increased by this factor.

From Appendix A, the cross-section for a process is defined such that $\sigma_{AB}\, v_a$ gives the rate of formation of state B. Therefore

$$\sigma_{AB} = \frac{\pi\hbar^2}{m_a^2 v_a^2} \cdot \frac{(2J_C+1)}{(2J_a+1)(2J_A+1)} \cdot \frac{\Gamma_A\Gamma_B}{(E_A-E_C)^2+\Gamma^2/4}, \tag{6.6}$$

and the factor in the first brackets can be written more familiarly as $\pi\lambda\!\!\!^{-2}$. In this expression E_A is the energy of the incoming beam of particles (in the centre-of-mass frame) and is therefore the energy variable. The shape of the resonance curve is shown in Fig.6.2, where the variation of $\pi\lambda\!\!\!^{-2}$ over the resonance has been neglected.

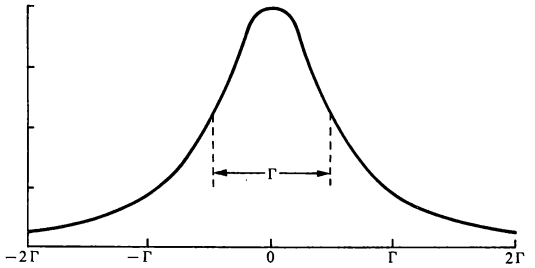

Fig.6.2. Shape of Breit-Wigner resonance denominator plotted against $(E-E_r)$, with energy units of Γ. If the resonance is sharp so that neither $\lambda\!\!\!^{-2}$ nor the widths vary appreciably over an energy range $\sim 4\Gamma$, then this curve will represent the behaviour of the reaction cross-section.

In the derivation of the Breit-Wigner formula above it has been assumed that the decay of C* is characteristic of C* alone and does not depend upon the way C* was produced. This independence of formation and decay of the compound nucleus was first stressed by Bohr. It is an important point, but has been somewhat overworked in the textbooks. It holds only for an isolated resonance, but in tests quoted

different formation processes can populate the overlapping resonances to different degrees, and the decay processes will then differ. If the number of overlapping resonances is very large then a statistical averaging process will reduce the difference. Thus independence of formation and decay is good for a single resonance or a large number of overlapping resonance, but probably does not hold for a few overlapping resonances.

Two outstanding examples of resonant reactions are shown in Figs. 6.3 and 6.4. In the (p,γ) reaction the resonances are sharp because the proton width is small due to the effect of the potential barrier. From the above, Γ is proportional to the square of

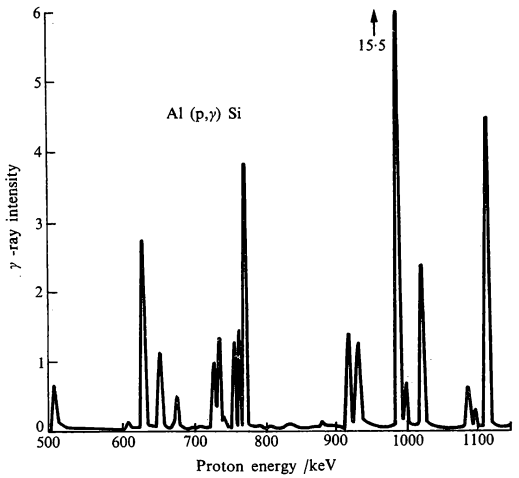

Fig.6.3. Resonances in ^{27}Al (p,γ) ^{28}Si. The resonance widths indicated represent experimental limitations; the true widths are considerably less. (Based on BROSTROM *et al.* (1947). *Phys.Rev.* 71, 661.)

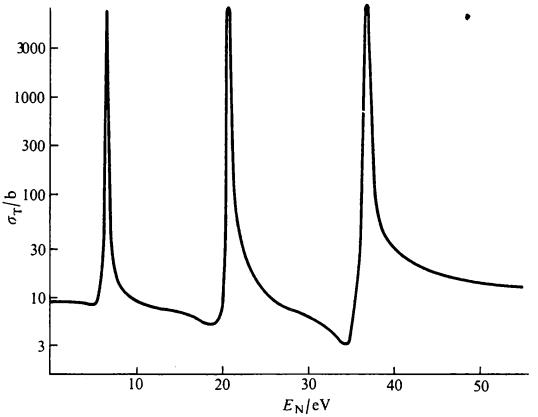

Fig.6.4. Resonances in $^{238}U(n,\gamma)^{239}U$ for neutrons up to 50 eV. Note the large cross-sections at resonance, the high density of levels in the compound nucleus ($\sim$ 1 every 15 eV) and the narrow widths. Note also that the measurements were made in absorption. Off-peak the absorption is due to 'hard-sphere' scattering interfering with the scattering from tails of the resonances; on-peak the absorption is dominated by the capture process.

the modulus of a matrix element which connects a state representing a particle within the nucleus with a state representing the same particle outside the nucleus (a plane wave at infinity) and the overlap of this latter wavefunction with the nucleus introduces the barrier-penetrability term. It is then surprising that the neutron resonances shown have even smaller neutron widths. This is partly accounted for by the fact that the separation between resonances has become small because in the example shown the target nucleus in (n,γ) is a heavy one and in (p,γ) a light one. But

why should crowding of levels result in smaller partial widths? This question can be conveniently answered by considering what would happen to cross-sections were it not the case. Assume that the cross-section is being averaged over an energy range ΔE large compared with resonance widths, then the contribution to the *total* reaction cross-section of a single resonance is obtained by integrating the Breit-Wigner expression (with Γ for Γ_B) giving a result proportional to Γ_A only. The number of levels in ΔE is $\sim \Delta E/D$, where D is the average spacing between levels. The averaged cross-section is therefore $\propto \Gamma_A/D$. As the energy is increased D will get smaller corresponding to an increase in level density with excitation of nucleus C. If Γ_A remains approximately constant with energy then the cross-section increases without limit. But cross-sections at high energies should approach something like the classical πR_N^2, and we are forced to conclude that (Γ_A/D) is roughly independent of energy. Thus the partial width into a given channel depends upon the local spacing of levels (see also Chapter 3). This does not mean that the total width of the resonance is similarly limited; it is a sum of all partial widths and each such partial width is proportional to the local spacing, but, as the energy increases, the number of open channels contributing also increases. Thus the total width of levels tends to increase with energy, but only slowly since the two effects almost compensate.

To return to (n,γ), it was stated that the above considerations only partly account for the small neutron widths. A closer look at the expression for the partial width in terms of the matrix element

(p.147) reveals that the width is proportional to the velocity in the channel and this is very small for neutrons of a few eV - so small that in this region $\Gamma_n < \Gamma_\gamma$. (This surprising circumstance is also a consequence of the fact that γ-radiation is taking place from a level at an excitation of $\sim$ 6 MeV or more in a large nucleus, so there is an extremely large number of available states of lower energy to which γ-transitions can take place.) In addition, the width includes the penetrability through the angular-momentum barrier (and for charged particles it is augmented by the Coulomb barrier) with the result that Γ_n in the eV region is far too small, when $l \neq 0$, to give observable resonances.

In the case of the (p,γ) resonances, their sharpness is determined largely by the barrier factor. The 'widths without barrier' in light nuclei are quite large fractions ($\sim$ a few per cent) of the single-particle proton width, but the barrier penetrability reduces them to a few eV giving a total width for the resonance less than the experimental resolution of the beam. The peak yield is not therefore the yield at resonance, but a measure of the integrated yield over the resonance.

Finally, the Breit-Wigner formula can be used to show that neutron cross-sections vary as $1/v$ at very low energies. So far it has been assumed that the Γs are constants in the formula but this is an oversimplification; the matrix elements can have an inbuilt energy-dependence and, as has been mentioned, there is a factor v_a in the expression for Γ_a. For a sharp resonance at high energy in the incident channel the variation is not expected to be large over

the region of large cross-section, but at very low energies the $\pi\lambda\!\!\!^{-2}$ term becomes large and enhances the low energy tail of the resonance. It is therefore necessary to consider the contribution made by a resonance far from the peak of the resonance denominator. The cross-section is then of the form $\sigma \propto \pi\lambda\!\!\!^{-2}\Gamma_n\Gamma_\gamma/(E_C)^2$, assuming that $E \ll E_C$ and also $\Gamma \ll E_C$. Now Γ_γ is unlikely to vary whilst the energy changes by a few eV since the level in C is 6 MeV or higher in excitation, so the energy variation is contained in $\lambda\!\!\!^{-2}\Gamma_n$, and since $\lambda\!\!\!^{-} \propto 1/v_n$ and $\Gamma_n \propto v_n$ then $\sigma \propto 1/v_n$.

Direct reactions

The characteristic of a compound-nucleus resonance is a cross-section which changes rapidly with energy over a small range of energy. The range over which the cross-section changes rapidly is of order Γ and is correlated with the time scale of the process by $\Gamma\tau \sim \hbar$. A direct reaction has been defined as one taking place whilst the bombarding particle passes through the nucleus and therefore in a characteristic time $\sim 10^{-22}$s. Using the above relationship, it is associated with a width $\sim$ 10 MeV. Thus a characteristic of direct reactions is that the cross-section should not change rapidly with energy, or rather should not exhibit sharp maxima, since it is possible for a direct reaction involving charged particles below the barrier to increase exponentially in cross-section.

Another characteristic concerns the angular distribution of the process. If a reaction proceeds via a single (and this must be emphasized), well-defined resonance then, at the intermediate stage of the process, it has a well-defined parity (the parity of the state C*). In setting up the state a definite direction has

been singled out (the direction of the beam), but this does not correspond to a well-defined parity, so in forming C* the information concerning direction of motion is lost but not the axis of the beam. (To form a state of well-defined parity it is necessary to superimpose upon the state defined by the beam, a corresponding state with the beam moving in the opposite direction.) Thus the subsequent decay of C* will give an angular distribution having symmetry about 90° (fore-aft symmetry). The direct reaction has no such restriction, the symmetry axis contains the information concerning direction and angular distributions do not in general have fore-aft symmetry. In fact there is a strong tendency to forward-peaking, an extreme example being the case of absorption of a fairly high-energy neutron beam which results in the diffractive scattering of the beam. As in the classical case of diffraction of light, the forward-directed scattering peak has angular width $\sim \lambda/R_N$ rad. Many other types of direct reaction conform to this simple diffractive picture.

The stripping reaction is an interesting example of a direct reaction. It is the reaction A(d,p)B (or equivalently (d,n)), in which one nucleon of the incoming deuteron interacts with the nucleus whilst the other passes it by. The probability of the process occurring depends upon the square of the modulus of $\int \psi_f^* V \psi_i \, d\tau$, where the initial wavefunction ψ_i is that of the deuteron in the form of a plane wave combined with its internal function, multiplied by the target wavefunction ψ_A, whilst the final wavefunction ψ_f will be that of the proton in the form of a plane wave multiplied by the wavefunction of the final nuclear

state ψ_B, which for simplicity will be taken as $\psi_A\psi_n$, where ψ_n is a single-particle wavefunction of the neutron about the core A. Notice that the Coulomb distortion of the plane waves has been neglected here, though it is included in the analysis of research data. For the perturbation V we can put in the interaction between the neutron and the nucleus A, which is taken to be of the form $V(r_n)$ with the centre of A defined as the centre of the coordinate system. It is also assumed that any change of shape of the deuteron during its journey past the nucleus can be neglected and that reaction only occurs when the proton and neutron in the deuteron are at their common centre. Another simplifying assumption made here is that the reaction takes place only at the surface of nucleus A - this is not assumed in a more serious calculation.

With these approximations, in the integral, the internal wave-function of the deuteron becomes merely a constant (the probability that the proton and neutron should be found at its centre), and its plane wave becomes $\exp(i\underline{k}_d.\underline{r}_n)$, whilst that of the proton becomes $\exp(i\underline{k}_p.\underline{r}_n)$, since $\underline{r}_n = \underline{r}_p = \underline{r}_d$, by assumption. By the same token $V = V(R_N)$ = constant, as also is $\psi_A^*\psi_A$. In the integral $\underline{r}_n \equiv (R_N,\theta',\phi')$ and the integral becomes

$$\text{constant} \times \int\psi_n^*(\underline{r}_n)\exp(i\underline{k}_d.\underline{r}_n)\exp(-i\underline{k}_p.\underline{r}_n)d\Omega'.$$

If $\underline{k}_d - \underline{k}_p = \underline{q}$, and the direction of $\underline{q}$ is taken to define the z'-axis the integral $\propto \int Y^*_{l,m}(\theta',\phi')\exp(iqR_N\cos\theta')d\Omega'$, where the single-particle angular wavefunction has been inserted for the neutron, but its spin function has been neglected.

The integral over ϕ' requires that $m = 0$, giving

$$\int_{\cos\theta'=-1}^{\cos\theta'=+1} Y_{l,0}(\theta')\exp(iqR_N\cos\theta')\,d(\cos\theta').$$

To see how this behaves take the cases $l = 0$ and $l = 1$. The first integral, for $l = 0$, can be performed at sight, giving constant $\times$ $(\sin qR_N/qR_N)$, which, when the substitution $q^2 = (k_d - k_p)^2 + 4k_d k_p \sin(\theta/2)$ is made, gives a maximum at $\theta = 0$, followed by subsidiary maxima of reduced intensity. (The function $(\sin qR_N/qR_N)^2$ is recognizable as the diffraction function, so, provided $|\underline{k}_d - \underline{k}_p|R_N$ is less than π, the statement made is true. This is the case in most experimental applications.) Putting in $Y_{1,0} \propto \cos\theta$, for $l = 1$, gives an integral $\int_1^1 \exp(iqR_N c)\,c\,dc$. Integrating by parts gives $(\sin qR_N - qR_N \cos qR_N)/(qR_N)^2$ which, on squaring, is seen to give zero at $q = 0$, rising to a large first maximum, and then having subsequent lesser maxima. With the same proviso as for $l = 0$, the angular distribution will be non-zero at $\theta = 0$ (unless $k_d = k_p$ which is possible, for low E_d and a high Q-value, but not likely) rising to a large first maximum (at a fairly small angle in practical cases) followed by subsidiary maxima. The important point is that the location of the primary maximum can be used to determine the l-value of the transfer, i.e. the amount of orbital angular momentum taken in by the stripped nucleon. Thus the parity of the final state is known immediately on analysis of the angular distribution of the particle not captured, and the range of J-value of the final state is limited. The degree of limitation depends upon J for the initial nucleus - if this

were zero then the final state must have $J_f = l_n \pm \frac{1}{2}$, where l_n is the transferred angular momentum. But in the general case J_f can take the range $(J_i + l_n + \frac{1}{2})$ down to the smaller of $|J_i - l_n \pm \frac{1}{2}|$. Examples of angular distributions in (d,p) are given in Fig.6.5.

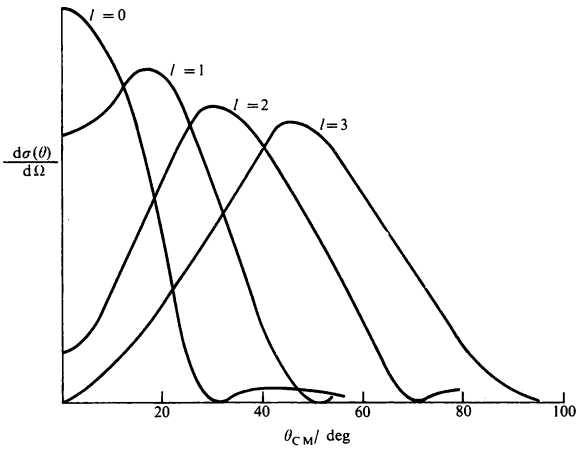

Fig.6.5. Typical angular distributions for the (d,p) direct reaction. If each process represented the full single-particle transfer, then relative to $l = 0$, the curves for $l = 1,2,3$ should be multiplied respectively by $\frac{1}{4}$, $\frac{1}{12}$, $\frac{1}{30}$.

A useful feature of the stripping reaction is that the magnitude of the cross-section is a measure of the degree to which the final state looks like the original state plus a single particle of the type indicated by the angular distribution. Of especial importance is the fact that the final state may be bound to the particle transferred (indeed the calcula-

tion is simpler for such bound states, and the simple assumptions made here would not be good for an unbound state). Because the transfer is virtual rather than real, the real positive momentum q can be associated with a negative total energy of transfer.

The stripping reaction is a typical direct reaction, of which there are also many other examples. The inverse (p,d) or pick-up reaction, (^{3}He, d), (α,p), etc., and their inverses are further examples of stripping. Other direct reactions are elastic and inelastic scattering of charged particles (usually α-particles) off distorted nuclei; Coulomb excitation - the electromagnetic excitation of a nucleus by the rapidly changing field of a passing charged particle; transfer reactions, e.g. (^{17}O, ^{16}O), in which a neutron is transferred from the projectile to the target - this is really an example of stripping with heavy-ion projectiles.

PROBLEMS

6.1. For elastic scattering of a projectile of mass m by a target of mass M show that the angle of scattering in the laboratory system (θ) and the centre-of-mass system (ϕ) are connected by $\phi = \theta + \sin^{-1}\{(m/M)\sin\theta\}$ and that the ratio of solid angles between the two systems is given by $d\Omega_{CM}^{(\phi)}/d\Omega_{lab}^{(\theta)} = \sin^2\phi/\sin^2\theta \cos(\phi-\theta)$. Explain the significance of these transformations in the measurement of differential cross-sections.

6.2*. A foil of $^7_3$Li is bombarded with protons of 5·0 MeV, and it is found that at 90° neutrons of 2·3 MeV are emitted. If the atomic masses of p,

n, and ^{7}Li are respectively 1·0078U, 1·0087U, and 7·0160U find the atomic number and mass of the daughter nucleus.

6.3*. A thin target of ^{27}Al is bombarded with 8 MeV protons, and the energy spectrum of the protons scattered at 45^{o} to the incident beam contains strong peaks at 7·84 MeV and 6·99 MeV. What can be deduced about the levels of this nucleus?

6.4. By using an argument involving parity, show that $J = \frac{1}{2}$ for the intermediate state in a γ-γ cascade will, like $J = 0$, result in an isotropic angular correlation. Show also that the γ-decay of an isomeric $J = \frac{1}{2}$ state will be isotropic even when the state has been completely polarized into one of its sub-states (e.g. by application of a magnetic field at low temperature), but will not be isotropic in this latter condition if the γ-detector is sensitive to the state of circular polarization.

6.5*. In a given nucleus, a state C (0^{+}) decays by γ-cascade through a state B to the ground state A (0^{+}). If the two γ-transitions are E1, what is the spin and parity of B? Will there be a non-isotropic angular correlation of the two γ-rays? Alternatively, if the γ-radiations are again E1 but B is known to be 0^{+}, what are the spins and parities of A and C? Will there be an angular correlation?

For the more general case, discuss qualitatively how angular-correlation measurements can help to give information about spins and parities of

nuclear states (a) if the multipole order of each γ-ray is known (e.g. from internal conversion measurements), (b) if it is not.

6.6*. Suppose that the reaction $^{12}C(\alpha,\gamma)\,^{16}O$ (Q = 7·15 MeV) shows an isolated resonance at E_α = 10·10 MeV with Γ_{tot} = 0·20 MeV, Γ_α = 0·15 MeV, and Γ_γ = 10 eV. The spin of the compound state is J = 4. The only other decay mode of ^{16}O at this excitation is by proton emission to the ground state ($J = \frac{1}{2}$) of ^{15}N. When the same level is excited by $^{15}N(p,\gamma)^{16}O$ (Q = 12·11 MeV) what will the peak cross-section be? (Remember to consider centre-of-mass effects.)

6.7*. (a) Write an equation embodying the principle of detailed balance as it applies to two-body reactions with unpolarized beams and targets. (This should be deduced from Fermi's golden rule and the definition of cross-section - since these have been used to deduce the Breit-Wigner formula it can also be conveniently extracted from the latter, though it has greater generality.) Explain how it may be used to deduce the spin of the pion from the measured cross-sections for $p + p \rightleftharpoons d + \pi^+$. (Note there is in addition a factor arising from the production of two identical particles when the reaction proceeds from right to left.) (b) Explain why the Breit-Wigner formula is important in the theory of slow neutron capture by nuclei. Show that, at a resonance peak (neglecting any background), $\pi\sigma^2 = \lambda^2 g\sigma_s$, where σ and σ_s are the total and

elastic scattering cross-sections for neutrons.

7. Fission and Fusion

In these days of energy crises no textbook on nuclear physics would be complete without some reference to these two sources of energy. Fusion has very little overlap with nuclear physics, except for the fact that the basic reactions are between light nuclei, but the nuclear reactor based on fission can truly be said to be the brain-child of the nuclear physicist. Indeed virtually the whole of nuclear physics is involved in describing how and why a reactor works.

THE FISSION PROCESS

In Chapter 1, the mass formula indicated that all nuclei above $A \sim 150$ are in principle unstable to α-emission. At the same time, the question of stability against splitting up in any way whatsoever might have been considered, but was not. It is so now. Splitting into two approximately equal parts is expected to be possible at large enough mass values, since the mass formula indicates that medium-weight nuclei are most stable. Expressing $M(Z,A) - M(Z_1,A_1) - M(Z_2,A_2)$, where $Z = Z_1 + Z_2$, $A = A_1 + A_2$, in terms of the mass formula (p. 18), and maximizing this function, gives the result $Z_1 = Z_2$, $A_1 = A_2$, and

$$M(Z,A) - 2M(\tfrac{1}{2}Z,\tfrac{1}{2}A) = -\beta A^{2/3}(2^{1/3}-1) + \varepsilon Z^2 A^{-1/3}(1-2^{-2/3}),$$

neglecting the trivial term in δ. The condition for stability is therefore

$$\frac{Z^2}{A} < \frac{\beta}{\varepsilon}\left(\frac{2^{1/3}-1}{1-2^{-2/3}}\right) \sim 0{\cdot}7\left(\frac{\beta}{\varepsilon}\right) \sim 17 \ .$$

Thus, in principle, all nuclei above $A = 90$ ($Z = 40$) are unstable to fission into equal fragments. This low value is reminiscent of the low value obtained for α-decay, which in turn leads to the reason why such light elements do not fragment, namely, the Coulomb barrier. It might be more realistic to expect fission to occur if the mass of the nucleus under consideration exceeds the masses of the two parts by more than the Coulomb energy when the two parts have just separated. Fission will then occur over the top of its barrier. This adds a term

$$\frac{(Ze/2)^2}{4\pi\varepsilon_0 . 2R_0 (A/2)^{1/3}}$$

to the energy of the two fragments. Inserting a suitable value for R_0 (1·3 fm), or alternatively assuming that ε derives from a uniform charge distribution and using the value of ε to determine the radius, modifies the condition to $Z^2/A > 50$ for instability, and this would correspond to an extrapolation to $Z \sim 130$ along the valley of stability.

These estimates are very crude. It has been assumed that, in inverse, as they move toward each other the two fragments remain spherical up to contact. But nuclei in isolation can be non-spherical, and even if spherical it is not realistic to assume they are rigid. In the presence of each other it is possible that the Coulomb potential be less than the monopole term predicts because of induced distortion or because the proton distribution inside

each nucleus may no longer be uniform. These and other mechanisms can diminish the Coulomb potential near the point of contact whilst not greatly affecting the potential of the single large nucleus. It is to be expected that spontaneous fission will occur for less massive nuclei than predicted by the above formula.

To consider the mechanism in more detail, a potential function is needed to describe the interaction between the two fragments. When the two components are distinct, the potential can be expressed as a function of the distance between their centres, but this parameter has little meaning before separation. In this region a convenient parameter is ΔR for an ellipsoidal nucleus as defined in Chapter 1 (p.31), but this ceases to be useful near the point of break-up since $\Delta R \to \infty$ produces a needle-like nucleus rather than two globules. If we ignore this difficulty the potential is shown in Fig.7.1, where the two dotted curves correspond to the limiting cases of a nucleus in principle unstable to fission but not observed to decay so and of a nucleus which would decay by the fission process in a time of order of the nuclear characteristic time. The interesting case is shown by the full line which represents a typical distorted nucleus in the region of uranium.

If the well representing the large nucleus is extremely shallow then it is possible for states inside it to decay by penetrating the fission barrier (since the particle penetrating has large effective mass, the penetrability varies extremely rapidly with energy below the top of the barrier so that shallow means of the order of a few hundred keV). This leads

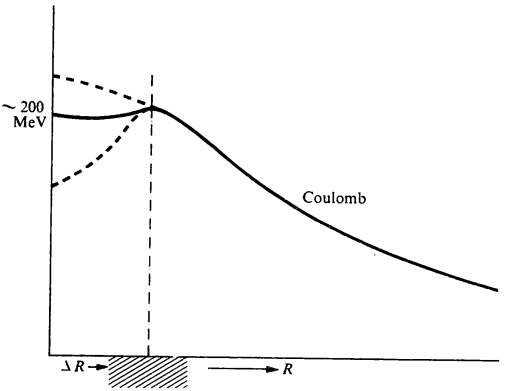

Fig.7.1. Schematic fission potentials. Note that the abscissa is the distortion parameter ΔR up to somewhere below separation and the distance between fragment centres above separation. The dashed curves are explained in the text.

to a class of radioactivity not previously discussed, namely, decay by spontaneous fission. If the well is deeper than this, then the levels near ground may be stable to this mode of decay, but levels of higher excitation will not be, so fission will take place in preference to γ-emission. Such levels could be excited by inelastic scattering, e.g. (n,n') or (p,p'). At a greater well-depth, the location of the top of the fission barrier relative to neutron instability is of great importance. If the fission barrier lies lower then the nucleus one neutron less than the one here considered, on absorbing a thermal neutron, will decay by fission; if the fission barrier

lies say 1 MeV higher then neutrons of energy greater than 1 MeV will be necessary to induce fission. Notice that a neutron threshold is well defined, but the fission threshold is effective in that just below it the process can take place but with very much reduced probability.

As an example of the above, spontaneous fission occurs in the following:

$^{235}_{92}U$ ($\tau_f \sim 2 \times 10^{17}$years), $^{238}_{92}U$ ($\tau_f \sim 10^{16}$years,

$^{240}_{94}Pu$ ($\tau_f \sim 10^{11}$years), $^{250}_{98}Cf$ ($\tau_f \sim 10^{4}$years),

$^{254}_{98}Cf$ ($\tau_f \sim$ 55days).

Nuclei which undergo fission on capture of thermal neutrons are ^{233}U, ^{235}U, and ^{239}Pu, whilst nuclei which require capture of fast neutrons are, with effective neutron thresholds in brackets, $^{232}_{90}Th$ (1·3 MeV), ^{234}U (0·4 MeV), ^{236}U (0·8 MeV), ^{238}U (1·1 MeV), and $^{237}_{93}Np$ (0·4 MeV). The two last sequences reflect the importance of even or odd neutron numbers, emphasizing the effect of pairing of nucleons.

For nuclei in this region the fission reaction is highly exothermic with a Q-value of order 200 MeV. This is highly desirable for production of energy, but of itself does not lead to power production. The reaction must be self-sustaining as in chemical combustion when part of the energy liberated in the reaction is used to heat the fuel and so encourage further reaction. From information now to be dis-

closed, the fission reaction can be made self-sustaining. Fission is not merely the splitting of a large nucleus into two smaller ones but, as is to be expected, a lot of debris is produced at the same time. Neutrons form part of this debris, either produced simultaneously with the large fragments or more probably by secondary emission from the fragments which will in general be formed in states of high excitation. In addition, the fragments lie on the neutron-rich side of the valley of stability (since N/Z increases with Z along the valley) and therefore have low binding energies for neutrons. These neutrons will be emitted almost instantaneously ($\tau \sim 10^{-16}$s or less) with a spectrum peaked at $\sim$ 1 MeV. This latter point can be understood in terms of the discussion on widths in Chapter 6; Γ_n from a given state to any final nuclear state is proportional to k or $E_n^{\frac{1}{2}}$, but the density of final nuclear states has very roughly an exponential dependence upon energy of excitation. The most probable energy of emission of a neutron is therefore the maximum of $(E-E^*)^{\frac{1}{2}}\exp(\alpha E^*)$, where E is the energy of the neutron transition to ground and E^* the excitation of the final nucleus. At maximum $E-E^* = 1/2\alpha$. Peaking at $E_n = E-E^* \sim 0{\cdot}7$ MeV gives $\alpha = 0{\cdot}7$ in units of $(\text{MeV})^{-1}$, indicating how the level density behaves for a heavy nucleus. Note that $\exp(\alpha E^*)$ is an oversimplification, but it does represent roughly the crowding together of levels at higher energies; α varies slowly from nucleus to nucleus.

In addition to these 'prompt' neutrons there are also emitted some delayed neutrons amounting to $\sim 0{\cdot}5$ per cent of all neutrons produced. These were

initially a source of mystery, but are now known to occur following β-emission; an example is the decay of the fission fragment $^{87}_{35}Br$ which β-decays with 55s half-life to $^{87}_{36}Kr$, with an available energy ∿ 6 MeV. As well as decaying to nucleon stable levels in ^{87}Kr, a fraction of the β-transitions are to states of excitation greater than 5·4 MeV, and these can decay by neutron-emission. Neutron-emission will therefore be delayed by the time taken for β-decay of ^{87}Br. Although relatively few, these delayed neutrons play a large part in the control of a reactor. If each neutron-induced fission event produces on average more than one neutron, the possibility of a self-sustaining chain reaction arises. Typically the number ν, per fission (average) is something like 2-2·5.

Before going on to discuss how to make use of the fission reaction, comment should be made on the reference to the fragment $A = 87$. This is a long way from the symmetric splitting ($A \sim 115$) which has been suggested as giving the greatest energy release. In fact the most probable mode of fission is not the symmetric mode, but corresponds roughly to an $A = 90$ and 140 split. The explanation of this is still required; magic numbers probably play a part since $N = 50$ for $A \sim 90$, $N = 82$ for $A \sim 140$. Note that the Coulomb term is less for the asymmetric split relative to the symmetric, offsetting to some extent the mass balance which favours the symmetric split. Prediction of the 'mass spectrum' requires detailed knowledge of the fission process - any mechanism which can change the fission barrier height, as a function of the fragment masses, by ∿ 0·1 MeV in 200 MeV could account for the spectrum. It is inter-

esting to note that, close to threshold, the (90, 140) splitting predominates and there is virtually no symmetric splitting. As the energy of the neutron inducing fission rises, the splitting becomes more nearly symmetric in accordance with the rough guidelines above.

THE CHAIN REACTION AND REACTORS

If a material fissile to fast neutrons produces ν neutrons per fission (on average) and the probability that, on absorbing a neutron (and the definition can be extended to include absorption in other materials present), fission occurs is p (again on average) then, if $p\nu > 1$, a chain reaction will occur in a large mass of this material, culminating in an explosion. The intensity of the explosion depends on the rate of liberation of energy compared with the rate at which energy can be transmitted away (possibly taking the material with it). If the interaction length for fission is λ then the interaction time is $\lambda/\bar{v}$, and in that (mean) time the neutron multiplication is $(p\nu\text{-}1)$, giving an exponential growth in neutron flux with time constant $\lambda/\bar{v}(p\nu\text{-}1)$. For ^{235}U, $\sigma_f \sim 1b$, giving an interaction length $\sim$ 20 cm, $\bar{v} \sim 10^9 cm\ s^{-1}$ ($E_n \sim 1$ MeV), and for $p\nu\text{-}1 \sim 1$ this gives a time constant $\sim 10^{-8}s$. For such a rapid rate of neutron growth and therefore energy production, heat dissipation is negligible, and because of inertia the material has no time to get away. The result is the so-called 'atom bomb'. From what has been said the critical size for explosion is something greater than 20 cm, and this can be achieved by imploding (chemically) some smaller blocks of ^{235}U.

On the face of it, it seems unlikely that such a system could be controlled. The fissile material could be diluted and $p\nu$ could be made very close to unity, but it would be difficult to produce just the correct conditions for a slow enough growth rate in order to be able to introduce some form of mechanical control, and if by error conditions are changed to make $p\nu$ slightly greater then the system runs amok. At this point the fact that $\sim$ 0·5 per cent of the neutrons are delayed becomes crucial; if $p\nu$ is just less than unity for prompt neutrons and just greater than unity for prompt-plus-delayed neutrons then the time constant of the delayed neutrons takes control and the growth-rate time constant will be considerably greater than this (since $p\nu$-1 $\sim$ 0·005). Thus there is a range of variation of $p\nu$ of $\sim$ 0·5 per cent over which the neutron flux is increasing at a controllable rate. For the fast reactor, which is effectively being discussed here, control would be achieved by moving parts of the fissile material away from the rest or by moving some of the surrounding material relative to the fissile core, thereby altering the number of neutrons being reflected back into the core.

Historically the thermal reactor preceded the fast reactor. Why was this the case? The answer lies in understanding what makes a suitable fuel. Although ^{238}U and some other nuclei listed on p. 167 are fissionable above a neutron threshold they are not suitable fuels for a fast reactor, whereas nuclei like ^{235}U, which are fissionable to thermal neutrons, are. For these latter nuclei the factor p depends only upon the ratio $\sigma(n,\gamma)/\sigma(n,f)$, since $\sigma(n,n')$, which is larger than $\sigma(n,f)$ at MeV

energies, does not remove neutrons but merely changes their energies. However, if ^{238}U were used as a fuel then (n,n') would rapidly remove neutrons below the fission threshold and so be akin to absorption, with a reduction in p. This process is enhanced by the fact that the reactor has to have a structure and a coolant so the neutron spectrum inside a fast reactor is degraded to lower energies very quickly. It is therefore necessary that the fuel be fissile down to low energies. Thus natural uranium, containing only 0·7 per cent ^{235}U, cannot be made to go critical; the fuel must be highly enriched in ^{235}U, a costly undertaking.

It would therefore seem surprising that any structure containing natural uranium can be made to go critical. If, however, we look at the various cross-sections at thermal energies ($\sim$ 0·025 eV) we find that, for ^{235}U, $\sigma(n,\gamma) \sim 101$ b and $\sigma(n,f) \sim 577$ b, whereas for ^{238}U, $\sigma(n,\gamma) \sim 2{\cdot}76$ and $\sigma(n,f) \sim 0$. Even taking into account that there is 140 times more ^{238}U than ^{235}U in natural uranium, a value $p \sim 0{\cdot}54$ is obtained for thermal neutrons. Since $\nu \sim 2{\cdot}44$ for ^{235}U, it is possible to obtain criticality provided that not more than a quarter of the neutrons are lost in thermalizing them. Before considering the thermalization process it is pertinent to ask why $\sigma(n,\gamma)$ for ^{238}U turns out to be so much smaller at thermal energies than the ^{235}U cross-sections. This is partly a fortunate accident, but not completely so.

On comparing the excitation functions of U + n for the different isotopes, an odd-even effect is immediately apparent. The even isotopes produce narrow, well-separated resonances having very high

peak cross-sections; the odd isotopes produce broader resonances, poorly separated (the density of resonances is ~ 10 times greater for the odd than the even isotopes) and with lower peak cross-sections. Previously it has been stated that level density increases exponentially with excitation, and the odd-even effect does ensure that the excitations produced by absorption in an odd isotope are about 1·5 MeV higher than in an even isotope reflecting the fact that in the former case the resulting ground state has one more pair of neutrons than the initial ground state ($^{235}U + n_{th}$ gives 6·48 MeV excitation, $^{238}U + n_{th}$ gives 4·78 MeV). But, because pairs must be split in order to create excited states there is an odd-even effect needed in applying the exponential function. It is probably more correct to translate energies such that neutron thresholds coincide in comparing level densities of neighbouring isotopes. There is, however, a $(2J+1)$ factor appearing in the level-density function, and this can account for the observed ratio, since $^{235}U + n$ produces states of $J^{\pi} = 3^-$ and 4^- (^{235}U ground state is $\frac{7^-}{2}$ and $l_n = 0$ for neutrons in the eV range) whilst $^{238}U + n$ produces $J^{\pi} = \frac{1}{2}^+$ only. Merely increasing the level-density ratio should not (on average) enhance the capture cross-section ratio at thermal energies, since $\sigma_{n\gamma} \propto \Gamma_n\Gamma_\gamma/E_r^2$ (assuming that $E_r > \Gamma$ - see the Breit-Wigner expression, p.148), and this can be written

$$\frac{(\Gamma_n/D)(\Gamma_\gamma/D)}{(E_r/D)^2}$$

Provided the top two factors remain much the same

from one isotope to the other it would appear that the thermal cross-section depends upon how close is the nearest level compared to the level spacing - and this is how chance creeps in. However, though (Γ_n/D) is not expected to vary much between neighbouring nuclei since it refers to a single channel; the same is not true for (Γ_γ/D), because it represents a summation over a large number of γ-transitions. The same $(2J+1)$ factor, which produced a greater density of resonances in $^{235}U + n$, also provides a greater density of states to which γ-transitions can be made since strong γ-transitions change J only by one unit at most. It therefore turns out that total radiation widths Γ_γ, following neutron absorption, tend to be similar for neighbouring nuclei, rather than Γ_γ/D. There is therefore a tendency for even-A isotopes to have lower thermal-capture cross-sections than odd-A isotopes, apart from any odd-even effects which may also occur because of pairing. This argument is little affected by the presence or absence of fission width, though it may need slight modification if the increased total width no longer satisfies $E_r > \Gamma$.

Returning to thermalization, neutrons always lose energy in elastic collisions, and the lighter the scattering nucleus the greater the loss. Hydrogen would therefore seem to be the most suitable material, conveniently perhaps in the form of water; but, unfortunately it has a relatively high thermal-neutron capture cross-section ($\sim$ 330 mb), and so absorbs many of the neutrons on slowing them down. It cannot be used with natural uranium, but uranium enriched in ^{235}U can go critical in a light (ordinary) water environment, and modern power reactors have been con-

structed to this basic design. The next suitable material is heavy water since ^{2}H is the next best thermalizer, and, moreover, $\sigma(n,\gamma)$ is very low (0·5 mb). Again, modern power reactors have been built using natural uranium fuel and heavy water. The next most convenient material is carbon, which is much poorer at moderating but still has a small enough $\sigma(n,\gamma)$ to be of value (3·4 mb). The early experimental reactors were built with carbon (graphite) as moderator and natural fuel, and modern power reactors have been so based.

The geometrical arrangement of fuel and moderator is important. The obvious homogeneous mixture is unsuitable for natural fuel since the neutron capture process in ^{238}U is important right down to its lowest resonance at $E_n \sim 6{\cdot}6$ eV (see Fig.6.4, p.151). Separation of the two processes of fission and moderation is needed, but the next obvious step of having all the fuel at the centre surrounded by moderator is also unsuitable. At thermal energies the effective cross-section for removal of neutrons is $2{\cdot}7 + (101 + 577)/140 \approx 7{\cdot}5$ b, weighting the cross-sections according to the isotopic abundance, giving an absorption length of order 2 cm. Thus the thermal-neutron flux into the fissile centre is strongly attenuated, and most of the fuel is inefficiently used. The solution is therefore a matrix of natural fuel rods a few millimetres in diameter embedded in the moderator, through which must also be circulated a coolant with low neutron-absorbing properties to transfer the heat produced to the outside world. At least this is the case for the graphite-moderated reactors, the suitable coolant being carbon dioxide

gas; the liquid moderators can, in addition, be used for heat transfer. Note that, because the fission fragments have such a short range, virtually all the heat is produced within the fuel itself, with a small fraction arising from slowing down the neutrons and absorption of γ-radiation.

The control of a thermal reactor derives once again from the existence of delayed neutrons, though the time constant when critical to prompt neutrons is rather slower because the neutrons, when thermalized diffuse rather slowly through the moderator (thermal speeds are $\sim 10^5$ cm s^{-1} so diffusion speeds are lower and distances of traverse to get back to fuel are $\sim$ 10 cm giving a (prompt critical) rise time $\sim$ ms). The mechanical control is in the form of cadmium rods that can be inserted into the reactor. Cadmium is a very strong absorber of thermal neutrons by virtue of the large value of $\sigma(n,\gamma)$ ($\sim$ 20,000 b) for the isotope $^{113}_{48}Cd$, of abundance 12·3 per cent. This high cross-section arises from a resonance very close to thermal energies, as has been discussed previously for ^{235}U.

So far only the rare isotope ^{235}U has been utilized as a nuclear fuel. Obviously if ^{238}U could also be used resources of power are considerably increased. This can be done, making use of the very process that has proved troublesome in the design of reactors discussed, namely, the capture process to form ^{239}U. ^{239}U undergoes two quick β^--decays to ^{239}Pu, which is long-lived ($\sim$ 24,000 years) and has already been listed among nuclei which are thermally fissile. This breeding of nuclear fuel can be best achieved with a fast reactor since ν tends to rise

with neutron energy and the greater its value the greater the surplus of neutrons for breeding over and above the requirements for sustaining the reaction. This surplus is made use of by making the core small enough to allow a large leakage of neutrons into a surrounding uranium blanket. Since the values of $\rho\nu$ are 2·45 and 2·70 for pure ^{235}U and ^{239}Pu, even after reasonable loss incurred from other materials and coolant, within the core, it is possible to breed more new fuel than is consumed. It is necessary, of course, to remove the uranium blanket from time to time to separate out chemically the ^{239}Pu, which is behaving effectively as a catalyst in the fission of ^{238}U. Thus, in principle, all of natural uranium can be usefully consumed and at $\sim$ 200 MeV per nucleus this works out at about 1g per day per MW!

FUSION IN THE SUN

Before discussing the nuclear physics behind attempts to produce fusion in the laboratory, it is of interest to describe briefly the nuclear reactions going on inside the nearest working fusion reactor, the Sun. Prior to the discovery of nuclear reactions the Sun's source of energy was a mystery. The rate of emission of radiant energy from the Sun is well known and so also is its mass; gravitational collapse of this mass provides a large source of energy, but nowhere near large enough to keep the Sun radiating for a few thousand million years, and there is geological evidence on Earth to indicate that it has done so for a considerable fraction of that time. On the other hand, gravitational collapse to the present size from a tenuous gas will provide enough energy to heat

the Sun up to a temperature $\sim 10^7$ K even allowing for radiation losses in the slow process. At that temperature, corresponding to a mean thermal energy $\sim$ 1 keV, protons are beginning to collide sufficiently violently with each other to penetrate the Coulomb barrier. The stumbling block to the onset of nuclear reactions culminating in the synthesis of helium from hydrogen would appear to be the initial reaction of two colliding protons. If we assume that two protons momentarily make a compound nucleus, the diproton, then this nucleus will decay back into the entrance channel thereby producing no reaction. However, a parallel though improbable decay is by β^+-emission to the deuteron. Improbable though it may be, the Sun contains a lot of colliding protons which cannot disappear by any other process, so this one must occur (though it has never been observed in the laboratory) initiating the sequence:

$$p + p \rightarrow d + e^+ + \nu$$

$$p + d \rightarrow {}^3He + \gamma$$

$${}^3He + {}^3He \rightarrow {}^4He + 2p$$

Taking the first two reactions twice, the nett result is

$$4p \rightarrow {}^4He + 2e^+ + 2\nu + Q, \qquad (7.1)$$

and from the mass values this liberates $\sim$ 7 MeV per proton absorbed (by comparison fission gives less than 1 MeV per nucleon involved). The choice of reactions after the first is dictated by the preponderance of

protons. Thus the possible third reaction $d + {}^3He \rightarrow {}^4He + p$ will not occur since all deuterons formed will be mopped up by the second reaction before they can get near a ^3He nucleus. On the other hand, the $^3He + p$ interaction can only decay back into the same channel, so ^3He is stable in the presence of protons and will build up in concentration until the third reaction takes place at the appropriate rate for equilibrium.

The rate at which a reaction proceeds depends upon $n_1(v_1).n_2(v_2).\sigma(v_{12}).v_{12}$, where n_1 and n_2 are velocity distributions of densities of particles 1 and 2 per unit energy range and v_{12} is their relative velocity. The two Maxwellian distributions can be combined into a single such function in terms of the reduced mass and relative velocity giving (dropping the suffixes) $n_1(v_1)n_2(v_2)v_{12} \propto v^3 \exp(-mv^2/2kT)$. $\sigma(v)$ can be written as $\sigma_0.P(E).(E_B/E)$, where $P(E)$ is the Gamow factor, $\exp(-2\pi Z_1 Z_2 e^2/4\pi\varepsilon_0 v)$, in the penetrability for the limiting case of an energy E well below the barrier height E_B, and therefore for a nuclear radius which is very small compared to the classical distance of closest approach. (E_B/E) includes the $\pi\lambda^2$ term in the cross-section. Expressed in this way σ_0 is the cross-section at $E \sim E_B$ and is of the order of 0·1 b for a decay by particle emission and perhaps $\sim$ 10 μb for decay by γ-emission.

The reaction rate is obtained by integrating over all v. The two key factors will be the rapidly falling $\exp(-E/E_T)$, where E_T, the thermal energy, is $\sim$ 1 keV at 10^7 K, and the rapidly rising $\exp(-\beta E^{-\frac{1}{2}})$ where, for $Z_1 = Z_2 = 1$ and E in keV, $\beta \sim 30$. Maximizing the product of these two functions (the rate of variation

of other factors can be neglected) gives $E^{3/2} \sim \beta E_T/2$, giving $E \sim 6$ keV. Thus in the region around 6 keV there will be a sharp peak in the reaction rate (see Fig.7.2). At this energy the penetrability factor is $\sim \exp(-12)$ or 10^{-5} and the cross-section for $p + d \rightarrow {}^3He + \gamma$ will be $\sim 10^{-10}$ b; the reaction

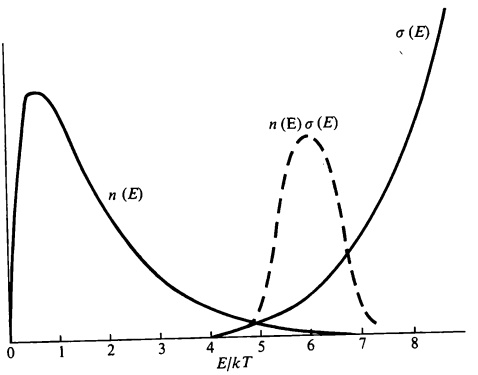

Fig.7.2. An indication of the peaking of the yield curve of a thermonuclear reaction.

${}^3He + {}^3He \rightarrow {}^4He + 2p$, having $Z_1 = Z_2 = 2$, will peak at ~ 15 keV and give a cross-section $\sim 10^{-10}$ b. For the p + p reaction the cross-section will in addition contain a factor representing the probability of β^+-decay during a nuclear characteristic time. A β^+-lifetime $\sim 10^3$s is a rough guess (about the same as the β^--decay of the neutron), and a nuclear time $\sim 10^{-21}$s gives the factor $\sim 10^{-24}$ and a cross-section at peak of about 10^{-29} b. Notice that none of these cross-sections is measurable in the laboratory; measurements could be pushed down to ~ 30 keV in

the second and ∿ 100 keV in the third reaction and the energy-dependence used to extrapolate further down. In the first reaction the factor 10^{-24} precludes measurement at any energy so the cross-section must be derived theoretically.

An alternative set of reactions to the above is the carbon-nitrogen cycle:

$$\begin{aligned} &{}^{12}C + p \rightarrow {}^{13}N + \gamma \quad ; \quad {}^{13}N \rightarrow {}^{13}C + e^{+} + \nu \\ &{}^{13}C + p \rightarrow {}^{14}N + \gamma \quad ; \quad {}^{14}N + p \rightarrow {}^{15}O + \gamma \\ &{}^{15}O \rightarrow {}^{15}N + e^{+} + \nu \quad ; \quad {}^{15}N + p \rightarrow {}^{12}C + {}^{4}He \ . \end{aligned} \tag{7.2}$$

Thus the ^{12}C is regenerated and acts as a catalyst. The importance of the two cycles depends upon the temperature and upon the amount of ^{12}C present. (Presumably the amount present in the primordial gas from which the Sun condensed, since formation of heavier elements requires a considerable concentration of ^{4}He in order to start the process with $3{}^{4}He \rightarrow {}^{12}C + \gamma$. This reaction is necessary since $p + {}^{4}He$ merely returns to the same channel, as does $^{4}He + He^{4}$.) At high enough temperatures (more massive stars) this latter cycle will take over, since none of the processes is particularly inhibited except for the Gamow factor, which will be larger because of the increased charges on C and N.

FUSION IN THE LABORATORY

Neither of the cycles described are suitable bases for terrestrially controlled fusion. The hydrogen cycle begins with the improbable (weak) re-

action of two protons whilst the carbon-nitrogen cycle requires higher temperatures which must be maintained over periods long compared with the 10 min life-time for β^+-decay of ^{13}N. The chief problems are the heating up of the reacting materials, and, more difficult, the containment of the hot material whilst sufficient reaction occurs to make a nett liberation of energy. A gas at 10^7 K (or strictly a plasma, since it will be highly ionized) cannot simply be retained by walls since, no matter what the material of the container, it will not be able to withstand bombardment of particles at such energies either physically or chemically. Much work has been carried out using 'magnetic bottles'; since the particles in the plasma are charged they are deflected by magnetic fields, and a great deal of cunning has gone into designing the field so that the particles are kept away from the walls of the vessel. In fact the magnetic fields can be used to feed energy into the plasma and to compress it into a smaller and smaller volume thereby increasing the reaction rate both by temperature and density increase. Unfortunately, the system becomes unstable, and so containment cannot be maintained indefinitely. Containment times are at present measured in fractions of a second, which of course can be large enough to liberate enormous amounts of energy as in the atom bomb or its later development the hydrogen bomb, which is merely an atom bomb surrounded by fusible material; the atom bomb raises the temperature high enough ($\sim 10^9$ K) and fusion then occurs more rapidly than the energy and the material can be dissipated.

Obviously the reaction, or cycle of reactions,

must be intrinsically fast. A suitable one for the laboratory would be

$$\begin{aligned} d + d &\rightarrow {}^{3}He + n \\ &\rightarrow T + p, \end{aligned} \tag{7.3}$$

since it uses deuterium only which occurs as 1/7000 of natural hydrogen and so is available in unlimited supply. The reaction involves unit charges and will therefore go at the lowest necessary temperatures (10^7 K in the Sun, but probably 10^8 K for short containment times). These two reactions have rather low Q-values (+ 3·27 MeV and + 4·03 MeV), releasing only $\sim$ 1 MeV per nucleon. In addition, they have relatively low peak cross-sections, $\sim$ 100 mb peaking at several hundred keV. On the other hand, the secondary reactions which could occur, namely $d + {}^{3}He \rightarrow p + {}^{4}He$ and $d + T \rightarrow n + {}^{4}He$, have higher Q-values (18·4 MeV and 17·58 MeV), and higher peak cross-sections, especially the latter which peaks at $\sim$ 80 keV with $\sigma \sim 7$ b (see Fig.7.3). This latter reaction therefore, would, go very quickly at 10^8 K, corresponding to a mean thermal energy of $\sim$ 10 keV, at which energy the cross-section is still as high as 10 mb. This mopping up, if complete, would boost the energy output to $\sim$ 4 MeV per nucleon.

From the above, the most convenient reaction to solve the problem of obtaining a useful source by fusion is undoubtedly d + T but it may be difficult to use it as a basis for large-scale energy production. In principle, the tritium could be regenerated by the reaction

$$n + d \rightarrow T + Y \ (6{\cdot}26 \text{ MeV})$$

or more conveniently by

$$n + {}^{6}Li \rightarrow T + {}^{4}He + 4{\cdot}78 \text{ MeV}.$$

This latter reaction has a high thermal cross-section, so that ^{6}Li in the form of a solid compound could form a suitable regenerating blanket, as in some fusion bombs. There is no

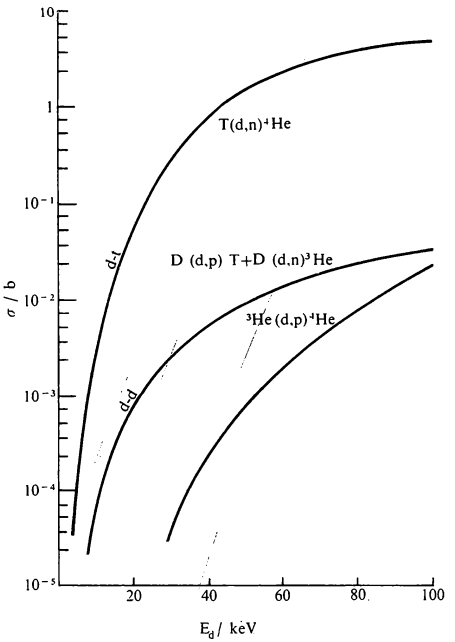

Fig.7.3. Thermonuclear reaction cross-sections. (Based on SEGRE (1964). *Nuclei and particles*, Benjamin, New York).

possibility, with these reactions, of breeding more tritium than is consumed but, by addition of Beryllium into the blanket, neutron multiplication can result from the dominant reaction

$$^{9}\mathrm{Be} + \mathrm{n} \rightarrow \alpha + \alpha + 2\mathrm{n} - 1{\cdot}66\ \mathrm{MeV},$$

leading to the possibility of tritium breeding.

A great deal depends upon the supply of materials (assuming that the technical problems are solved). The abundances in the earth's crust in parts per million are, for some of the relevant elements:

H, 1400(^{2}H, 0·2); Li, 65(^{6}Li, 4·8); Be, 6;
and compare with U, 4 (^{235}U, 0·03).

These figures do not give a complete picture, since hydrogen is readily available from water (the sea) and deuterium separation is probably the simplest of all isotope separations. Unless the breeding of tritium is relatively simple, it would seem that tritium-induced fusion will only be a stepping stone towards the final solution based on the d + d reaction.

PROBLEMS

7.1. If neutrons (mass m) of kinetic energy E_0 are isotropically scattered by nuclei of mass M show that the scattered neutron energy-spectrum is independent of energy within the limits E_0 and $\alpha^2 E_0$, where $\alpha = (M-m)/(M+m)$. Show further that

the mean value of $\ln (E_0/E_1)$, where E_1 is the energy after a single scatter, is equal to $1 + (a \ln a)/(1 - a)$, denoted by ξ.

7.2. Extend the above result to the case where the initial neutrons are not well defined in energy but have a spectral distribution. Hence show that, after n collisions, $\overline{\ln (E_0/E_n)} = n\xi$. If the mean energy of the neutron fission spectrum is 2 MeV, show that $\sim$ 18, 25, and 117 collisions are required to thermalize neutrons in H, D, and ^{12}C respectively.

7.3. An intimate mixture of ^{235}U and graphite is required for certain experiments. The graphite is known to be contaminated with 1 p.p.m. by weight of ^{10}B. What is the maximum fraction by weight of ^{235}U in the mixture if the multiplication factor at infinite size is not to exceed unity? (σ_{abs} = 0·04 b, 3800 b, and 700 b respectively for ^{12}C, ^{10}B, and ^{235}U at thermal energies; of the last cross-section, σ_f accounts for 580 b. Assume 2·5 neutrons per fission and that all reactions take place at thermal energies only.)

7.4. For a gas consisting of two types of particles, show that the distribution of *relative* velocities of pairs consisting of one from each type is the same as the absolute velocity distribution for a single type of particle but with a mass equal to the reduced mass, $m_1m_2/(m_1 + m_2)$, where m_1, m_2 are the masses of the two types.

7.5. (a) If the fusion reaction $d + d \rightarrow {}^{3}He + n + 3{\cdot}2$ MeV takes place with deuterons at rest, what is the kinetic energy of the neutron? (b) If the deuterons must come within 10^{-11} cm of each other, what energy must be supplied to overcome the electrostatic repulsion? (c) If this energy is supplied by heating the deuterium to high temperature, what order of magnitude of temperature is required? (d) Discuss the effect of this heating on the energies of the neutrons produced.

7.6. A nucleus consists of a large number N of particles of spin $\frac{1}{2}$. Determine the number of ways, $R(M_S)$, a given M_S can be produced assuming that $M_S \ll \frac{1}{2}N$ and that the Pauli Principle can be ignored. By identifying the difference $R(X) - R(X+1)$ with the number of states of total spin $S = X$, show that this number $\propto (2X+1)$. Hence the plausibility of the statement made on p.173.

Appendix A: Cross-sections

To introduce the concept of cross-section, it is convenient to consider classically a nuclear reaction in which a beam of particles falls on nuclei which behave like completely absorbing spheres of cross-sectional area σ. Assume that one such nucleus is confined somewhere within an aperture of area A through which passes a uniform beam of n particles per second. The number absorbed per second will obviously be $n\sigma/A$. If other nuclei are successively added to build up a practical target of thickness t and nuclear number density ρ (i.e. the number of nuclei per unit volume of target material), then ρAt nuclei will be exposed to the beam, and the number of incident particles absorbed per second will be $\rho At n\sigma/A = \sigma n\rho t$, provided that $\sigma\rho t \ll 1$. (If this condition does not hold then the beam flux is decreasing through the target and all nuclei are not exposed to the same beam - under these circumstances the beam decreases exponentially through the target, in the form $n(t) = n(0)\exp(-\sigma\rho t)$). Notice that the result does not depend upon the aperture area A since, no matter what this area is, provided that the target area is larger, each particle in the beam passes through the same target thickness.

Thus, for the absorption process the number per second absorbed is proportional to the cross-sectional area of the nucleus. Replacing this completely absorbing nucleus by a hard sphere of the same size which scatters all projectiles hitting it, the total number per second scattered out of the beam will be the number

per second previously absorbed, $\sigma n \rho t$. The number per second scattered into solid angle $d\Omega$ at (θ,ϕ), in the usual nomenclature, relative to the beam will be $\sigma n \rho t f(\theta,\phi) d\Omega$, where $\int f(\theta,\phi) d\Omega = 1$, to ensure that the total scattering has the correct value. $f(\theta,\phi)$ is known as the angular distribution of the nuclear process. It is convenient to combine the product $\sigma f(\theta,\phi)$ into a single function $d\sigma(\theta,\phi)/d\Omega$, which is known as the differential cross-section for scattering along the direction (θ,ϕ). Most nuclear reactions are observed under conditions which are characterized by an axis of symmetry in the direction of the beam - so, in general, $\sigma(\theta,\phi)$ becomes $\sigma(\theta)$. Experiments may be carried out under conditions in which two axes are defined, e.g. the beam axis and an axis of polarization, and for these the generalized direction (θ,ϕ) is required.

These simple pictures can be extended to cover more complex situations: the nucleus may not be completely absorbing or scattering but partially the one, partially the other, and partially transparent; and, following absorption, something is usually emitted - the same, or another, type of particle or γ-radiation. These processes can all be included in a general definition of cross-section provided the equivalence of cross-section and nuclear cross-sectional area is revoked. This equivalence has arisen merely from the simple picture used to introduce the concept, and will in any case require modification when the wave nature of the projectile is taken into account. The fact remains, however, that the intrinsic nuclear property which determines the yield of a reaction has the dimensions of an area.

The general definition of cross-section for the reaction A + a → B + b is contained in the equation

$$\frac{dY_{a,b}(\theta)}{d\Omega}\, d\Omega = n\rho t\, \frac{d\sigma_{a,b}(\theta)}{d\Omega}\, d\Omega \quad , \qquad (A.1)$$

where $dY_{a,b}(\theta)/d\Omega$ is the number of particles of type b per second emitted into unit solid angle at an angle θ to the beam, and $d\sigma_{a,b}(\theta)/d\Omega$ is the differential cross-section for the process (a,b).

Also of use is the partial cross-section for the process (a,b) defined as

$$\sigma_{a,b} = \int \frac{d\sigma_{a,b}(\theta)}{d\Omega}\, d\Omega$$

The total cross-section is defined as $\sigma_{a,tot} = \sum_b \sigma_{a,b}$, where the summation is carried out over all possible reaction products B + b including the incident channel (known as elastic scattering). This latter cross-section is very useful when the projectiles are neutrons, but not so for charged particles, when it is dominated by the comparatively uninteresting but very large Coulomb cross-section.

Finally, notice that cross-section has been defined in terms of the flux of incoming particles that is their density times their velocity. In calculating reaction rates using perturbation theory the incoming particle wavefunction is normalized to represent one particle per unit volume, so in going to the cross-section a factor v is needed; σv is a measure of the reaction rate as calculated by perturbation theory. If the incoming particle is just above threshold, it may be possible that the appropriate matrix element

varies only slowly with energy, as will also the density of final states if the reaction has positive Q-value. Under these circumstances $\sigma v \sim$ constant or $\sigma \propto 1/v$. This has been deduced in the text from the compound-nucleus formula, but as is seen above it has greater generality.

Appendix B: The need for a pairing term in the mass formula

In order to illustrate the need for a pairing term in the mass formula, it is convenient to neglect all terms in the mass equation but the Coulomb and symmetry terms and to simplify these to the form

$$a(A-2Z)^2 + bZ(Z-1) \qquad (a,\ b \text{ both positive})$$

The conditions that (A,Z) is stable relative to its neighbours $(A,\ Z-1)$ and $(A,\ Z+1)$ is

$$a(A-2Z)^2 + bZ(Z-1) < a(A-2Z-2)^2 + b(Z+1)Z$$

and also

$$< a(A-2Z+2)^2 + b(Z-1)(Z-2)$$

These can be re-expressed as

$$b.2Z > 4a\{A-2Z-1\}$$

and

$$b.2Z < 4a\{(A-2Z-1) + (1 + \frac{N}{Z-1})\} \quad .$$

Adding a nucleon to (A,Z) gives:

adding p, the energy change $E_p = -a(2A-4Z-1) + b.2Z$,

adding n, the energy change $E_n = +a(2A-4Z+1)$.

The condition for $(A+1,Z)$ to be stable relative to

$(A+1,Z+1)$ is that $E_p > E_n$, which simplifies to

$$b.2Z > 4a\{(A-2Z-1) + 1\} .$$

For the second neutron to produce a stable nucleus, the condition is that

$$M(A+2,Z) < M(A+2,Z+1),$$

which is equivalent to increasing A to $A+1$ in the last inequality, i.e.

$$b.2Z > 4a\{(A-2Z-1)\}+ 2$$

These conditions may be expressed in terms of an allowable variation of the function $b.2Z/4a$ as:

for (A,Z) stable, the range is $0 \rightarrow 1 + \frac{N}{Z-1}$ from the value $(A-2Z-1)$,

for first neutron, the range is $1 \rightarrow 1 + \frac{N}{Z-1}$ from the value $(A-2Z-1)$,

for second neutron, the range is $2 \rightarrow 1 + \frac{N}{Z-1}$ from the value $(A-2Z-1)$.

Since N/Z varies from 1 for light nuclei to $\sim 1\frac{1}{2}$ for the heaviest nuclei, these ranges are, for the heavy nuclei $2\frac{1}{2}$ units, $1\frac{1}{2}$ units, and $\frac{1}{2}$ unit respectively. Thus the probability of a neutron adding is roughly $\frac{3}{5}$, and a further neutron roughly $\frac{1}{3}$. A similar simple argument with protons would lead to $\frac{2}{5}$ for the first proton and zero for the second. Using all the terms of the mass equation and the more exact functions for the two discussed does not alter the general con-

clusion that if (A,Z) is stable then the addition of two more nucleons is more likely to be of the form (p,n) for stability than (2p) or (2n). Thus odd-odd nuclei should be as prolific as even-even.

Obviously a term is missing from the mass equation. It must be such that if A and Z are both even then, whilst not altering the probability that the first particle be a neutron for stability ($\frac{3}{5}$ is just about correct for a heavy nucleus), it must increase the probability for a second neutron giving stability up to unity. This extra term is not needed in considering the stability of odd-A nuclei, but must be chosen such that in going from even A to $A+2$, where both nuclei are stable, pairs of like particles will be added. Hence the form adopted.

Appendix C: Exchange forces

To introduce this concept it is interesting to consider the single ionized H_2 molecule - a system of two protons and one electron. When the separation

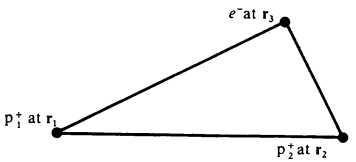

Fig.C.1. Schematic representation of the hydrogen-molecule ion.

$|\underline{R}| = |\underline{r}_2 - \underline{r}_1|$ is very large the electron wavefunction will look like the hydrogen-atom bound state situated at $\underline{r}_1$ or $\underline{r}_2$ and designated by $\phi(\underline{r}_3-\underline{r}_1)$ or $\phi(\underline{r}_3-\underline{r}_2)$. However, the correct wavefunction must be symmetric (or antisymmetric) with respect to the two protons. Therefore, put $\psi = \alpha_{\pm}^{-\frac{1}{2}}\{\phi(\underline{r}_3-\underline{r}_1) \pm \phi(\underline{r}_3-\underline{r}_2)\}$, where the normalization integral α takes the value 2 as $R \to \infty$, whilst $\alpha_+ \to 4$, $\alpha_- \to 0$ as $R \to 0$. The energy integral is

$$4\pi\varepsilon_0 \frac{V}{e^2} - \frac{1}{R} = -\int\psi^*\left(\frac{1}{|\underline{r}_3-\underline{r}_1|} + \frac{1}{|\underline{r}_3-\underline{r}_2|}\right)\psi d\tau_3 =$$

$$= -2\int\frac{\psi^*\psi}{|\underline{r}_3-\underline{r}_1|}\, d\tau_3 ,$$

by symmetry. Substituting for ϕ gives four integrals, and for ϕ real (corresponding to a bound state) two are the same giving (where $\underline{r} = \underline{r}_3-\underline{r}_1$)

$$4\pi\varepsilon_0 \frac{V}{e^2} - \frac{1}{R} = -2\alpha_{\pm}^{-1}\{\int\phi^2(\underline{r})\frac{1}{r}\,d\tau + \int\phi^2(\underline{r})\,\frac{1}{|\underline{r}-\underline{R}|}\,d\tau \pm$$

$$\pm\, 2\int\phi(\underline{r}-\underline{R})\phi(\underline{r})\frac{1}{r}\,d\tau\}. \quad (C.1)$$

The first term represents the binding energy of a hydrogen atom, the second term represents the electronic contribution to the Coulomb energy of a proton distant R from a hydrogen atom; for large R this term exactly cancels the proton repulsion ($1/R$ on the left-hand-side), but for small R it becomes the finite binding energy of the hydrogen atom. The last term, unlike the other two, has no classical analogue; it effectively arises from not knowing to which proton the electron belongs. Its sign depends upon the overlap of the two atomic wavefunctions when separated by R and, if the wavefunctions have nodes, can be positive or negative. For the lowest s-state there are no nodes, so the integral is always positive, giving increased binding for the symmetric wavefunction and reduced for the antisymmetric. At large distances bound wavefunctions fall off exponentially. If the s-state looks like $\exp(-\kappa r)$ at large r, the main contribution to the integral will occur nearly halfway between the two protons and will obviously contain $\exp(-\kappa R)$ in the product of the wavefunctions (this qualitative argument neglects the effects of terms like $1/r$ in the integral). Thus the integral will also behave like $\exp(-\kappa R)$ at large distances.

Now assume that the existence of the electron is not known, but that experiments can be performed to determine the potential between a proton (charged) and a hydrogen atom (neutral) as a function of separation R. The classical term will be constant at large R (but

unknown since the structure of hydrogen is unknown); however, at small R, the proton gets inside the electron cloud and experiences a net repulsion by the other proton. For the symmetric case, the non-classical term increases exponentially (approximately) to a finite binding at $R = 0$. The state of affairs is presented graphically in Fig.C.2.

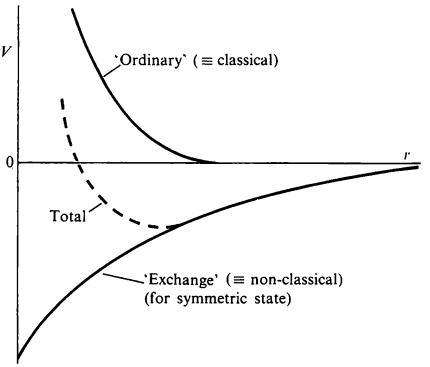

Fig.C.2. Potential of hydrogen-molecule ion. V=0 corresponds to infinite separation between ionized atom (p) and neutral atom (H).

Thus there eixsts a potential between the proton and the hydrogen atom which depends upon the spatial symmetry of the (p,H) system. During the interaction the proton and the hydrogen atom cannot be individually identified, and this is interpreted as exchange of charge and spin (in the form of an unknown electron). From Fig.C.2 this can result in attraction at large R, whilst the repulsion at small R arises because the 'fundamental-particle' hydrogen atom in fact has a structure.

The analogue with nucleons is obvious, at least for the (p,n) system. A similar argument to the above can be applied if n is looked upon as $p + \pi^-$ and gives the correct range for the nuclear force. Since the proton and neutron have almost the same mass, the total energy (rest + binding) of the bound π^- must be nearly zero. But $\kappa = |p|/\hbar$ in the unphysical region, where $|p|$ is given by $p^2c^2 = E^2 - m_\pi^2 c^4$ with $E = 0$. Therefore $\kappa = m_\pi c/\hbar = (1{\cdot}4 \text{ fm})^{-1}$. In order to produce charge-independence, a π^0 must be introduced (to give the (p,p) and (n,n) interactions) and also a π^+ (though its need is not immediately obvious; it arises because the above picture has made the proton look 'more fundamental' than the neutron - the π^+ redresses the balance since its exchange appears to make the proton look like $n + \pi^+$).

Having gone through the above reasoning, it is convenient to forget the mechanism and express the different exchanges mathematically. The operator $\underline{s}_1 \cdot \underline{s}_2 \equiv \frac{1}{2}\{(\underline{s}_1+\underline{s}_2)^2 - \underline{s}_1^2 - \underline{s}_2^2\} = \frac{1}{2}\{\underline{S}^2 - \underline{s}_1^2 - \underline{s}_2^2\}$ takes the value $\frac{1}{2}\{S(S+1) - s_1(s_1+1) - s_2(s_2+1)\}$ for the spin state S, resulting from the combination of two spin states s_1 and s_2. For $s_1 = s_2 = \frac{1}{2}$, it takes the value $+\frac{1}{4}$ for $S = 1$ and $-\frac{3}{4}$ for $S = 0$. Thus the operator $P^\sigma \equiv \frac{1}{2} + 2\underline{s}_1 \cdot \underline{s}_2 = 1$ for $S = 1$, triplet state, and $= -1$ for $S = 0$, singlet state. This P^σ is the spin-exchange operator. If $\uparrow\downarrow$ is used to denote the state in which particle 1 has spin up, particle 2 spin down, then it can be represented thus:

$$\uparrow\downarrow = \tfrac{1}{2}(\uparrow\downarrow + \downarrow\uparrow) + \tfrac{1}{2}(\uparrow\downarrow - \downarrow\uparrow) = \frac{1}{\sqrt{2}}\{(S=1) + (S=0)\},$$

therefore

$$P^{\sigma}\uparrow\downarrow = \frac{1}{\sqrt{2}}\{(S=1) - (S=0)\} = \tfrac{1}{2}(\uparrow\downarrow+\downarrow\uparrow) - \tfrac{1}{2}(\uparrow\downarrow-\downarrow\uparrow) = \downarrow\uparrow.$$

Thus P^{σ} has exchanged spins. P^{T}, which exchanges charge, can be defined similarly. The general potential is then (see Chapter 2, p. 40, for nomenclature) $V = V_W + V_H P^T + V_B P^{\sigma} + V_M P$ (where $P = -P^T P^{\sigma}$ is a requirement of the Pauli principle). Since the exchange operator has a purely numerical part, the potential can be rewritten

$$V = V_1 + V_2 \underline{t}_1\cdot\underline{t}_2 + V_3\underline{s}_1\cdot\underline{s}_2 + V_4(\underline{s}_1\cdot\underline{s}_2)(\underline{t}_1\cdot\underline{t}_2)$$

and the operators are (loosely) referred to as exchange operators.

The effect of the operators, P^{σ}, P^{T}, P^{S}, is shown in Table C.1 for S- and P- states. Notice that it is the pattern of the changes of sign which is important. The sign of a whole column can be changed merely by changing the sign of the potential function accompanying the operator.

TABLE C.1

ψ	Symmetry††			Sign of operator†			
	Spin	Isospin	Space	W	H	B	M
$^3S_1$	s	a	s	+	-	+	+
$^1S_0$	a	s	s	+	+	-	+
$^3P_{012}$	s	s	a	+	+	+	-
$^1P_1$	a	a	a	+	-	-	-

†see p.40 for definition of W,H,B,M.

††s denotes 'symmetric'; a denotes 'antisymmetric'.

Appendix D: Spin matrices

Angular momentum can be introduced into quantum mechanics by inserting the operator form $\underline{p} = -i\hbar\nabla$ into the classical expression $\underline{J} = \underline{r}\wedge\underline{p}$, to give $\underline{J} = -i\hbar(\underline{r}\wedge\nabla)$. It is left to the reader to show that $\underline{J}\wedge\underline{J} = i\hbar\underline{J}$ (hint: use Cartesian coordinates), i.e. $M_x M_y - M_y M_x = i\hbar M_z$ and two others, cyclically.

Although these equations have been derived from considerations of orbital angular momentum which is quantized to integer values, Dirac has used them to show that odd half-integer values are also permitted indicating the existence of another type of intrinsic angular momentum. Here it is proposed to consider $J = \frac{1}{2}$, and to designate it s, the intrinsic spin angular momentum of a fundamental particle. For this particular case it can be shown that $s_x s_y + s_y s_x = 0$, etc. and therefore $s_x s_y = (i\hbar/2)s_z$, etc. Since s_z is two-valued, having values $\pm\frac{1}{2}\hbar$, the wavefunction representing an arbitrary state of spin can be written as a two-component vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in an appropriate two-dimensional space. Ths components of $\underline{s}$, namely, s_x, s_y, s_z will be represented by 2×2 matrices, of which s_z will be diagonal and of the form $\frac{1}{2}\hbar\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to give the required eigenvalues when acting on the eigenfunctions $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ representing 'spin up' and 'spin down'. Putting $s_x \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $s_y \equiv \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, and using $s_y s_z = \frac{1}{2}i\hbar s_x$ and $s_z s_x = \frac{1}{2}i\hbar s_y$ results in

$$a = e = d = h = 0, \quad g = ic, \quad f = -ib.$$

There is some latitude in the choice of s_x, s_y; the forms commonly chosen are

$$s_x = \tfrac{1}{2}\hbar \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } s_y = \tfrac{1}{2}\hbar \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} .$$

Notice that $s_x^2 + s_y^2 + s_z^2 = s^2 = \frac{3}{4}\hbar^2$, which conforms to the usual expression $J^2 = \hbar^2 J(J+1)$.

The raising operator

$$\hbar^{-1}s^+ \equiv \hbar^{-1}(s_x + is_y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

obviously operates on $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to give $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and on $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to give $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$; hence its name. Similarly the lowering operator $\hbar^{-1}s^-$ is defined as $\hbar^{-1}(s_x - is_y)$.

In a similar way the components of isospin may be defined. As for spin, the two-component isospin space has no connection with any classical space, but, and in contrast to spin, neither does the three-component isospin space in which the components τ_x, τ_y, τ_z are defined. There is a one-to-one correspondence between the z-component of isospin and the charge of the nucleon, but otherwise there is no further definition of this space. The isospin raising and lowering operators have the property of changing one nucleon type to its counterpart, without changing any other part of its wavefunction. It should be stated that this is a rather restricted view of isospin since only nucleons are under consideration as distinct from its greater ramifications when dealing with other hadrons.

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Those indicated by an * are postgraduate in standard.

Answers to problems

1.1. $n \sim 3$.

1.4. $\varepsilon \sim 0{\cdot}63$, $\gamma \sim 19{\cdot}1$.

1.5. 3·7 fm.

2.2. + 0·31 nm.

2.5. $\sim$ 2·07 MeV.

3.4. $I \sim \frac{2}{3} I_{\text{rigid}}$.

3.6. $\lambda \sim 16$, 4·8, 6·3, 16 fm respectively.

4.4. $\sim$ 3 s.

5.1. $\sim 6 \times 10^{-18}$ s.

6.2. 7·0191 a.m.u.

6.6. 8·5 μb.

7.3. 0·00115.

7.5. (a) 2·4 MeV; (b) 14 keV in Centre-of-mass system; (c) $\sim 1{\cdot}6 \times 10^{8}$ K; (d) thermal spread $\sim \pm$ 100 keV.

Index

Physical constants and conversion factors

Avogadro constant	L or N_A	$6{\cdot}022 \times 10^{23}$ mol^{-1}
Bohr magneton	μ_B	$9{\cdot}274 \times 10^{-24}$ J T^{-1}
Bohr radius	a_0	$5{\cdot}292 \times 10^{-11}$ m
Boltzmann constant	k	$1{\cdot}381 \times 10^{-23}$ J K^{-1}
charge of an electron	e	$-1{\cdot}602 \times 10^{-19}$ C
Compton wavelength of electron	$\lambda_C = h/m_e c$	$= 2{\cdot}426 \times 10^{-12}$ m
Faraday constant	F	$9{\cdot}649 \times 10^{4}$ C mol^{-1}
fine structure constant	$\alpha = \mu_0 e^2 c/2h$	$= 7{\cdot}297 \times 10^{-3}$ ($\alpha^{-1} = 137{\cdot}0$)
gas constant	R	$8{\cdot}314$ J K^{-1} mol^{-1}
gravitational constant	G	$6{\cdot}673 \times 10^{-11}$ N m^2 kg^{-2}
nuclear magneton	μ_N	$5{\cdot}051 \times 10^{-27}$ J T^{-1}
permeability of a vacuum	μ_0	$4\pi \times 10^{-7}$ H m^{-1} exactly
permittivity of a vacuum	ϵ_0	$8{\cdot}854 \times 10^{-12}$ F m^{-1} ($1/4\pi\epsilon_0 = 8{\cdot}988 \times 10^{9}$ m F^{-1})
Planck constant	h	$6{\cdot}626 \times 10^{-34}$ J s
(Planck constant)/2π	$\hbar$	$1{\cdot}055 \times 10^{-34}$ J s $= 6{\cdot}582 \times 10^{-16}$ eV s
rest mass of electron	m_e	$9{\cdot}110 \times 10^{-31}$ kg $= 0{\cdot}511$ MeV/c^2
rest mass of proton	m_p	$1{\cdot}673 \times 10^{-27}$ kg $= 938{\cdot}3$ MeV/c^2
Rydberg constant	$R_\infty = \mu_0^2 m_e e^4 c^3/8h^3$	$= 1{\cdot}097 \times 10^{7}$ m^{-1}
speed of light in a vacuum	c	$2{\cdot}998 \times 10^{8}$ m s^{-1}
Stefan–Boltzmann constant	$\sigma = 2\pi^5 k^4/15h^3 c^2$	$= 5{\cdot}670 \times 10^{-8}$ W m^{-2} K^{-4}
unified atomic mass unit (^{12}C)	u	$1{\cdot}661 \times 10^{-27}$ kg $= 931{\cdot}5$ MeV/c^2
wavelength of a 1 eV photon		$1{\cdot}243 \times 10^{-6}$ m

1 Å $= 10^{-10}$ m; 1 dyne $= 10^{-5}$ N; 1 gauss (G) $= 10^{-4}$ tesla (T);
0°C $= 273{\cdot}15$ K; 1 curie (Ci) $= 3{\cdot}7 \times 10^{10}$ s^{-1};
1 J $= 10^{7}$ erg $= 6{\cdot}241 \times 10^{18}$ eV; 1 eV $= 1{\cdot}602 \times 10^{-19}$ J; 1 cal$_{th}$ $= 4{\cdot}184$ J;
ln 10 $= 2{\cdot}303$; ln $x = 2{\cdot}303 \log x$; e $= 2{\cdot}718$; log e $= 0{\cdot}4343$; $\pi = 3{\cdot}142$